Completeness theorems via the double dual functor

Adriana Galli, Gonzalo E. Reyes* and Marta Sagastume
Universidad de La Plata, *Université de Montréal

May 1998

Abstract: The aim of this paper is to apply properties of the double dual endofunctor on the category of bounded distributive lattices and some extensions thereof to obtain completeness of certain non-classical propositional logics in a unified way. In particulart, we obtain completeness theorems for Moisil calculus, n-valued Lukasiewicz calculus and Nelson calculus. Furthermore we show some conservativeness results by these methods.

The double dual functor has a long and distinguished history. It was used by E. Galois in the XIX century and by M. Stone in ours. One can even say that it appears, in some form at least, in the work of J-B. Fourier in the XVIII century. In Stone's work it originates through a duality between Boolean algebras and zero dimensional compact spaces, the so-called Stone spaces. The dual of a Boolean algebra is the set of Boolean morphisms between the algebra and the two-element Boolean algebra **2** with the Stone topology. Similarly, the dual of a Stone space is the Boolean algebra of the continuous maps between the space and the two-element discrete space **2**. Since **2** belongs to both categories, it is usually referred to as a 'schizophrenic' object.

A fundamental feature of this duality is its functorial character: the duality is extended to maps. Thus, the dual of a Boolean map between Boolean algebras is a continuous map between the corresponding topological spaces (although in the opposite direction). Similarly, the dual of a topological map between topological spaces is a Boolean map between the corresponding Boolean algebras (again in the opposite direction). Stone's paper was the starting point of related work on a large number of dualities: the Zariski duality between commutative rings and ringed topological spaces, the Pontryagin duality between abelian groups and compact abelian groups, the Lefschetz duality between vector spaces, the Grothendieck/Galois duality between k-algebras and profinite G-sets, etc. Several dualities related to Stone duality have appeared in logic. Probably the best-known among these is the Priestley duality between distributive lattices and compact totally order disconnected topological spaces. The reader may find some of these developments in [8].

In most of these dualities, the double dual of a structure is isomorphic to the structure itself. In fact, the word 'duality' is mainly used in this restricted sense. However, this need not happen in general: certainly the double dual of a vector space is not isomorphic to the original one, unless the original vector space has further properties, e.g., is finite-dimensional. We shall use the word 'duality' in this extended sense.

In this paper we follow [10] and exploit precisely the fact that the double dual of a structure may have more structure and may be better behaved than the original one. Thus, it may be profitably used to obtain information about the structure that we started with. We shall deal (as in [10]) with one basic duality, that between bounded distributive lattices and posets, and several of its enrichments. A basic fact is that the double dual of a bounded distributive lattice is a complete Heyting algebra, which is also a co-Heyting algebra (i.e. whose order-theoretic dual is a Heyting algebra). Such algebras are called bi-Heyting. Furthermore there is a natural map, the evaluation map, between the original lattice and its double dual having two remarkable properties. First it is one-to-one and secondly it is a lattice map which is conditionally bi-Heyting. In particular, if the original lattice is a Heyting (resp. co-Heyting, resp. bi-Heyting) algebra, the evaluation map preserves this further structure.

More generally, a basic property of the double dual construction is that if we enrich the original lattice with further structure, the double dual inherits this further structure in a large number of cases and the evaluation map preserves that added structure. By using the well-known connection between propositional logics and algebras due to Lindenbaum and Tarski, this results in completeness theorems. But this is not all: as its name indicates, the double dual functor is a functor and the double dual of a map preserves the richer structure that the double dual has. Furthermore, the evaluation map is a natural transformation. This allows us to obtain completeness theorems for propositional modal logics (as in [10]) and for propositional non-classical logics such as Nelson's constructive logic with strong negation, Łukasiewicz n-valued logic, etc. This way of obtaining these completeness theorems seems to be new.

In [10] these methods were 'lifted' to define double duals for coherent categories (which play the role of bounded distributive lattices among 'logical' categories) and some of their enrichments, thus obtaining completeness theorems for first order intuitionistic and modal logics. We have not investigated the corresponding extensions in our case.

This paper is organized as follows: in the first part we describe the basic duality between the category of bounded distributive lattices and the category of posets. One of the main tools is the known theorem that says that the evaluation map from a bounded distributive lattice is conditionally bi-Heyting (see e.g. [10]). Given the central role it plays in our paper we give a proof of this theorem. The second part is devoted to enrichments of (the category of) distributive lattices: de Morgan algebras, symmetric Heyting algebras, n-valued Łukasiewicz algebras and Nelson algebras. In each case we show that the double dual is an algebra of the same nature and that the evaluation map is a morphism of the corresponding category. Furthermore, the double dual of a map preserves the extra structure. Some of these results seem new. The last part deals with the applications to completeness theorems for different logics and conservation theorems between some of these. Roughly speaking we go from logic to algebra via the Lindenbaum-Tarski construction and then use the previous results. When interpreted in logical terms, these results yield the corresponding completeness theorems for the logics mentioned above.

Acknowledgements

The first and the last author would like to thank the Universidad de La Plata, for granting them a leave of absence, the ANPCYT and the FOMEC of Argentine and the NSERC of Canada (through a grant to Reyes) for their financial support, and the Pontificia Universidad Católica de Chile and the Université de Montréal for their hospitality. The second author would like to acknowledge the financial support of the Fundación Andes de Chile, the

FOMEC and the Universidad de La Plata, where most of this research was carried out during a sabbatical year (96/97) from the Université de Montréal. This year he spent as a guest of the Pontificia Universidad Católica de Chile. During all this time, he held a grant from NSERC (Canada). The support of all of these institutions is greatfully acknowledged. Finally, we would like to thank Marie La Palme Reyes for her help in editing this paper and the referee for a careful reading of this paper that resulted in several improvements.

1 The basic duality

This section deals with the duality between bounded distributive lattices and posets, following [10]. To help the reader along, we have listed mostly without proofs all the relevant results of that paper.

Let $\mathcal{D}l$ be the category of bounded (i.e. with 0 and 1) distributive lattices with lattice homomorphisms as morphisms. On the other hand, let $\mathcal{P}oset$ be the category of ordered sets with order preserving functions as morphisms. We let '**2**' denote both the two-element distributive lattice and the two element total order. (The 'schizophrenic' object mentioned in the Introduction.)

From now on, we shall use $\mathcal{D}l$ for both lattices and lattice morphisms. Similarly for other categories.

A morphism f in $\mathcal{D}l$ is *conservative* if it reflects the order: if $f(a) \leq f(b)$, then $a \leq b$. Notice that this boils down to f being a monomorphism, but order-reflection is a more basic property in our study.

We let $(\ )^* : \mathcal{D}l^{op} \longrightarrow \mathcal{P}oset$ to be the functor defined as follows: if $D \in \mathcal{D}l$, we let $D^* = \mathcal{D}l(D, \mathbf{2})$, i.e., the set of lattice morphisms $D \to \mathbf{2}$ ordered with the pointwise ordering inherited from $\mathbf{2}$. Furthermore, if $f : D_1 \longrightarrow D_2$, we let $f^* : D_2^* \longrightarrow D_1^*$ be the map $f^*(\ ) = (\ ) \circ f$, i.e., post-composing with f. Clearly, f^* is order-preserving.

In a similar vein, we define $(\ )^+ : \mathcal{P}oset^{op} \longrightarrow \mathcal{D}l$ by the following prescription: if $P \in \mathcal{P}oset$, let $P^+ = \mathcal{P}oset(P, \mathbf{2})$, namely the set of order-preserving maps $P \to \mathbf{2}$ with the point-wise lattice operations $\vee$ and $\wedge$ inherited from $\mathbf{2}$, 0 is the constant map 0 and 1 the constant map 1. If $f : P_1 \longrightarrow P_2$, we let $f^+ : P_2^+ \longrightarrow P_1^+$ be the map of post-composing with f: $f^+(\ ) = (\ ) \circ f$. It is clear that P^+ thus defined is a bounded distributive lattice and that f^+ is a lattice morphism.

Proposition 1
(1) Let $f, g : D_1 \longrightarrow D_2$ *be lattice morphisms such that* $f \leq g$*. Then* $f^* \leq g^*$.

(2) Let $f, g : P_1 \longrightarrow P_2$ *be order-preserving maps such that* $f \leq g$*. Then* $f^+ \leq g^+$.

Proof.
We shall prove the first only, the other being similar. Let $h \in D_2^*$ and $x \in D_1$. Then $f(x) \leq g(x)$. Since h is order-preserving, $h(f(x)) \leq h(g(x))$, i.e., $h \circ f \leq h \circ g$, since x is arbitrary. Thus, $f^*(h) \leq g^*(h)$ and this shows (since h is arbitrary) that $f^* \leq g^*$.□

Corollary 2 *The composite* $(\ )^{*+}$ *is a functor from the category* $\mathcal{D}l$ *into itself.*

The functor $(\ )^{*+}$ is called the *double dual.* Notice that $(\ )^{+*}$ is also a functor. We shall not consider it in this paper.

Remark 3 *Set-theoretically,* the above definitions and notions may be reformulated as follows: let $D \in \mathcal{D}l$. A morphism $h : D \rightarrow \mathbf{2}$ may be identified with a prime filter p of D: $\forall x \in D(x \in p \Leftrightarrow h(x) = 1)$. Thus, D^* becomes identified with the poset of all prime filters of D, with the set-theoretic inclusion as order. Furthermore, if $f : D_1 \longrightarrow D_2$ is a lattice morphism, $f^* : D_2^* \longrightarrow D_1^*$ is identified with the inverse image map f^{-1}.

The elements $f \in P^+$, on the other hand, may be identified either with the upward closed subsets (or up-sets) X of P by the prescription: $\forall p \in P(p \in X \Leftrightarrow f(p) = 1)$ or with the downward closed subsets (or down-sets) X of P by the prescription: $\forall p \in P(p \in X \Leftrightarrow f(p) = 0)$. We shall follow the first alternative.Thus, P^+ becomes identified with the lattice of up-sets of P ordered by set-inclusion. Similarly, if $f : P_1 \longrightarrow P_2$ is an order preserving map, $f^* : P_2^* \longrightarrow P_1^*$ becomes identified with the inverse image map f^{-1}.

For any poset P , P^+ is a complete lattice. In fact, D^{*+} is a complete Heyting algebra (see below).

The following will be useful in the sequel.

Proposition 4 *Let $f : B \longrightarrow A$ be a lattice morphism, B a Boolean algebra. If f is a monomorphism, then f^* is an epimorphism and f^{*+} a monomorphism.*

Proof.
Let $m \in B^*$ and let m_A be a maximal filter that contains $f(m)$. Then, $m \subseteq f^{-1}(m_A)$, which implies $m = f^{-1}(m_A)$ because m is maximal. This means that f^* is an epimorphism.

Let $U, V \in B^{*+}$ such that $f^{*+}(U) = f^{*+}(V)$. For $r \in U$ there exists some $q \in A^*$ such that $r = f^{-1}(q)$. But $f^{-1}(q) \in U$ if and only if $f^{-1}(q) \in V$. So, $U = V$.$\square$

Notice that we have a canonical *evaluation map*

$$e_D : D \longrightarrow D^{*+}$$

given by: $e_D(d)(\phi) = \phi(d)$ or, in set-theoretical formulation, $\phi \in e_D(d)$ iff $d \in \phi$.

Proposition 5 *The evaluation map $e_D : D \longrightarrow D^{*+}$ is a conservative lattice morphism.*

The following property of the evaluation map will be fundamental for us:

Proposition 6 *The family $e = (e_D)_{D \in \mathcal{D}l}$ is a natural transformation*

$$e : Id_{\mathcal{D}l} \longrightarrow (\ )^{*+}$$

in the sense that the diagram

$$\begin{array}{ccccc} & & e_{D_1} & & \\ & D_1 & \longrightarrow & D_1^{*+} & \\ f & \downarrow & & \downarrow & f^{*+} \\ & D_2 & \longrightarrow & D_2^{*+} & \\ & & e_{D_2} & & \end{array}$$

is commutative.

Proposition 7 *If $f : D_1 \longrightarrow D_2 \in \mathcal{D}l$, then $f^{*+} : D_1^{*+} \rightarrow D_2^{*+}$ has both a left and a right adjoint:*

$$\lhd \dashv f^{*+} \dashv \bigcirc$$

given explicitly (for $p \in D_1^*$ and $Y \in D_2^{*+}$) as follows:

$$p \in \lhd Y \Leftrightarrow \forall q \in D_2^*(f^*q \geq p \Rightarrow q \in Y)$$
$$p \in \bigcirc Y \Leftrightarrow \exists q \in D_2^*(f^*q \leq p \,\&\, q \in Y)$$

Let D be a bounded distributive lattice. If $x, y \in D$ the *implication* of x and y, $x \to y$ is the element (if it exists) determined uniquely by the property:

$$\forall z \in D \; (z \leq x \to y \Leftrightarrow x \wedge z \leq y)$$

The *negation of* x is defined by $\neg x = x \to 0$ (provided that $x \to 0$ exists).

Recall that a Heyting algebra is a bounded distributive lattice in which all implications exist.

The following notation will be useful in the next proof and elsewhere. If P is a poset and $Z \subseteq P$

$$(Z] = \{y \in P | \exists z \in Z \; y \leq z\}$$

$$[Z) = \{y \in P | \exists z \in Z \; y \geq z\}$$

Proposition 8 *For any poset P, the lattice P^+ is a complete Heyting algebra.*

Proof.
In fact, for $X, Y \in P^+$, we have:

$$x \in X \to Y \Leftrightarrow \forall y \geq x \; (y \in X \Rightarrow y \in Y)$$

Equivalently,

$$X \to Y = (X \cap Y^c]^c$$

where $(\;)^c$ denotes set-theoretical complementation.

Corollary 9 *The double dual D^{*+} of a bounded distributive lattice D is a Heyting algebra*

A morphism $h : A \to B \in \mathcal{D}l$ is *conditionally Heyting* if it preserves all existing implications: if $x, y \in A$, and $x \to y$ exists, then $h(x) \to h(y)$ exists in B and is equal to $h(x \to y)$.

Since the following result is fundamental for our paper, we will sketch a proof:

Theorem 10 *Let D be a bounded distributive lattice. Then the evaluation map*

$$e_D : D \to D^{*+}$$

is conditionally Heyting.

Proof.
We must show that :

$$e(d_1 \to d_2) = e(d_1) \to e(d_2)$$

provided that $d_1 \to d_2$ exists.

Assume that $d_1 \to d_2$ exists.

$\subseteq$: Since $e \in \mathcal{D}l$, $e(d_1 \to d_2) \wedge e(d_1) = e((d_1 \to d_2) \wedge d_1) \subseteq e(d_2)$. Thus, by definition $e(d_1 \to d_2) \subseteq e(d_1) \to e(d_2)$.

$\supseteq$: Suppose that $p \notin e(d_1 \to d_2)$, i.e., $d_1 \to d_2 \notin p$. We claim that there exists a (prime filter) $q \supseteq p$ such that ($q \in e(d_1)$ & $q \notin e(d_2)$). Thus, $d_1 \in q$ & $d_2 \notin q$.

Indeed, let $r = \{d : d \geq d_1 \wedge x, \quad x \in p\} = [p, d_1]$. Then r is a filter and $d_2 \notin r$ (otherwise $d_2 \geq d_1 \wedge x$. Therefore $x \leq d_1 \to d_2$ and $d_1 \to d_2 \in p$, a contradiction). By Zorn's lemma, there exists a filter q which is maximal among those with the following property: $q \supseteq r$ & $d_2 \notin q$. This q is a prime filter with the required properties. □

If A and B are Heyting algebras, a *Heyting morphism* $h : A \to B$ is a lattice morphism that preserves the Heyting implication. Notice that in this case, 'Heyting' and 'conditionally Heyting' coincide.

Proposition 11 *For a map $\phi : Q \to P$ of Poset a sufficient condition for $\phi^+ : P^+ \to Q^+$ to be a Heyting morphism is for ϕ to be upward surjective in the sense that for any $q \in Q$ and $p \in P$ with $p \geq \phi(q)$ there is $r \in Q$ with $r \geq q$ and $\phi(r) = p$.*

The following gives a sufficient condition for upward surjectivity:

Proposition 12 *For a Heyting morphism $h : A \to B$, $h^* : B^* \to A^*$ is upward surjective.*

Corollary 13 *If $h : A \to B$ is a Heyting morphism, then $h^{*+} : A^{*+} \to B^{*+}$ is Heyting.*

Let D be a bounded distributive lattice. If $x, y \in D$ the *difference* $x \setminus y$ is the element (if it exists) determined uniquely by the condition

$$x \setminus y \leq z \Leftrightarrow x \leq y \vee z$$

A *co-Heyting algebra* is a bounded distributive lattice such that any two elements have a difference. It follows that every element has a *supplement* $\sim x = 1 \setminus x$. Notice that, equivalently, a co-Heyting algebra is a bounded distributive lattice whose order-theoretic dual is a Heyting algebra (since the dual of the implication $x \to y$ is the difference $y \setminus x$ and the dual of $\neg$ is $\sim$.)

A *bi-Heyting algebra* is a bounded ditributive lattice that is both a Heyting and a co-Heyting algebra. Equivalently, a co-Heyting algebra is a Heyting algebra whose order-theoretic dual is again a Heyting algebra.

Theorem 14 *If P is a poset, P^+ is a complete bi-Heyting algebra.*

(1) The implication is given by

$$x \in X \to Y \Leftrightarrow \forall y \geq x \; (y \in X \Rightarrow y \in Y)$$

Equivalently,

$$X \to Y = (X \cap Y^c]^c$$

(2) The difference is given by

$$x \in Y \setminus X \Leftrightarrow \exists y \leq x \; (y \in Y \quad \& \quad y \notin X)$$

Equivalently

$$Y \setminus X = [X^c \cap Y)$$

A *conditionally co-Heyting* (respectively bi-Heyting) map is a lattice map which preserves all existing differences (respectively all existing implications and differences) of the domain lattices.

The following is a consequence of theorem 10 and duality:

Theorem 15 $e_D : D \to D^{*+}$ *is conditionally bi-Heyting.*

If A and B are co-Heyting (respectively bi-Heyting) algebras, a *co-Heyting morphism* (respectively a *bi-Heyting morphism*) $h : A \longrightarrow B$ is a lattice morphism which preserves all differences (respectively all differences and all implications) of the domain.

Notice that if A and B are co-Heyting, a conditionally co-Heyting morphism between them is the same as a co-Heyting morphism. Similarly for the Heyting case.

2 Enrichments of the basic duality

In this section we study some enrichments of the basic duality: symmetric Heyting algebras, n-valued Łukasiewicz algebras and Nelson algebras.

2.1 Symmetric Heyting algebras

A *De Morgan algebra* is a bounded distributive lattice together with a unary operator $(\ )'$ satisfying the following identities:

(M1) $(x \vee y)' = x' \wedge y'$

(M2) $x'' = x$

Thus, a De Morgan algebra is an algebra of the form $< A, \vee, \wedge, ', 0, 1 >$. In the following we deal with symmetric Heyting algebras (see [11]), also called De Morgan-Heyting algebras or briefly DH-algebras defined as follows:

An algebra $< A, \vee, \wedge, \longrightarrow, ', 0, 1 >$ is a *DH-algebra* if $< A, \vee, \wedge, ', 0, 1 >$ is a De Morgan algebra and $< A, \vee, \wedge, \longrightarrow, 0, 1 >$ is a Heyting algebra.

A *DH-morphism* is a lattice morphism which preserves the De Morgan and Heyting operators.

We let $\mathcal{DH}$ be the category of DH-algebras and DH-morphisms.

Remark 16 The difference $\setminus$ can be defined in every DH-algebra A. In fact, for all $x, y \in A$ $y \setminus x = (x' \longrightarrow y')'$ (see [11]). Thus, every DH-algebra is a bi-Heyting algebra.

For each DH-algebra A, $B(A)$ will denote the center of A, i.e., the subalgebra of all complemented or boolean elements of A. (Notice that this makes sense for any distributive lattice.)

For $x, y \in A$, $x \Longrightarrow y$ (respectively $y \setminus \setminus x$) will denote, in case it exists, the greatest boolean element z such that $z \wedge x \leq y$ (respectively the least boolean element z such that $z \vee x \geq y$). In particular, $1 \Longrightarrow x$ is the necessity operator $\Box(x)$ and $x \setminus \setminus 0$ is the possibility operator $\Diamond(x)$ (see [12], [6]).

For a fixed x of A, define the operators $I_x : B(A) \longrightarrow A$ by $I_x(z) = z \wedge x$ and $S_x(z) = z \vee x$. We let $R_x : A \longrightarrow B(A)$ to be the right adjoint of I_x (if it exists) and, similarly, $L_x : A \longrightarrow B(A)$ to be the left adjoint of S_x (if it exists). Thus

$$R_x(y) = x \Longrightarrow y$$

$$L_x(y) = y \setminus \setminus x$$

In particular, when $I_1 = S_0$ is the inclusion $i : B(A) \hookrightarrow A$ and the above adjoints exist, $R_1 = \Box$, $L_0 = \Diamond$. So we have $\Diamond \dashv i \dashv \Box$.

Remark 17 We use the symbol $\Diamond$ (respectively $\Box$) to denote both the function from A to $B(A)$ and the function from A to A.

Given a De Morgan algebra A the *dual structure* $< A^*, \subseteq, g_A >$ of A is defined by the set A^* ordered by inclusion and the map $g_A : A^* \longrightarrow A^*$ given by the prescription:

$$g_A(p) = A - \{x' : x \in p\}$$

The map g_A is called the *transformation of Birula-Rasiowa* (see [11]). It is an antiisomorphism of period 2.

On the other hand, let $(P, \leq)$ be a poset that is endowed with an antiisomorphism g of period 2. Then, $< P^+, \cup, \cap, n, 0, 1 >$ is a De Morgan algebra, where $n(X) = g(X)^c$, for $X \in P^+$.

Given a De Morgan algebra A, we call $< A^{*+}, \cup, \cap, n, 0, 1 >$ the *double dual structure* of A. By the above, it is also a De Morgan algebra.

Proposition 18 *If A is a De Morgan algebra, then the map $e_A : A \longrightarrow A^{*+}$ is a De Morgan morphism, i.e., for all $x \in A$, $e_A(x') = n(e_A(x))$.*

Proof.
Immediate from the condition: $y \notin g_A(p)$ if and only if $y' \in p$, which follows from the definition of g_A.□

Proposition 19 *Let $f : M \longrightarrow N$ be a De Morgan morphism. Then f^{*+} is a De Morgan morphism, i.e., $f^{*+}(n(X)) = n(f^{*+}(X))$.*

Proof.
It is easy to see that, for every $q \in N^*$, $g_M(f^{-1}(q)) = f^{-1}(g_N(q))$. Also, we have: $f^{*+}(n(X)) = \{q : g_M(f^{-1}(q)) \notin X\}$ and $n(f^{*+}(X)) = \{q : f^{-1}(g_N(q)) \notin X\}$. Then, $f^{*+}(n(X)) = n(f^{*+}(X))$. □

Theorem 20 *The double dual $(\ )^{*+}$ is an endofunctor on the category $\mathcal{DH}$. Furthermore, e is a natural transformation such that for every object A of $\mathcal{DH}$, e_A is conservative.*

Proof.
From [10] and the last two propositions.□

Lemma 21 *Let $< A, \vee, \wedge, \longrightarrow, ', 0, 1 >$ be a DH-algebra such that for every $x, y \in A$ there exist $x \Longrightarrow y$ and $y \setminus \setminus x$. Then,*

(1) $x \Longrightarrow y = \Box(x \longrightarrow y)$,

(2) $y \setminus \setminus x = \Diamond(y \setminus x)$,

(3) $\Diamond(x') = (\Box x)'$.

Proof.
Both terms in (1) are defined to be the largest $b \in B(A)$ such that $b \wedge x \leq y$. We deduce (3) from the definitions and (2) from (1) and (3).□

Remark 22 (1) There are some sufficient conditions for the existence of possibility and necessity operators. In [12], Reyes and Zolfaghari define the operators $\Diamond^n$ on a bi-Heyting algebra A as the composition: $(\sim \neg)^n$. In particular, in A^{*+} we have $\Diamond^n = [([(...])])$ with n times [(and n times]).

Dually, the operators $\Box^n$ are defined by: $\Box^n = (\neg \sim)^n$.

If A is ω-complete (that is, has supremum and infimum of countable subsets) then possibility and necessity operators are defined by the equalities $\Diamond x = \bigvee_{n \geq 1} \Diamond^n x$ and $\Box x = \bigwedge_{n \geq 1} \Box^n x$.

(2) Notice that if A is a DH-algebra in which: $\neg x \vee \neg\neg x = 1$ holds (Stone condition) then the Monteiro possibility operator $\triangle(x) = \neg\neg x$ (see [11]) coincides with Reyes-Zolfaghari operator $\diamondsuit$ because for each x, $\neg\neg x$ is the least boolean element greater than x.

Let $(P, \leq)$ be a poset. The *components* of P are the equivalence classes given by the equivalence relation generated by the relation of comparability. More precisely, let $\bowtie$ be the relation of comparability, i.e., $p \bowtie q$ iff $p \leq q$ or $p \geq q$. We define $p \bowtie_n q$ iff there are $p_1, p_2, \ldots, p_{2n+1}$ such that $p = p_1 \bowtie p_2 \bowtie p_3 \ldots \bowtie p_{2n+1} = q$. If for every pair p, q of elements in a component of P $p \bowtie_n q$ holds, then the component is *n-linked.* If every component of P is n-linked then P is *n-linked.*

Recall that $X \subseteq P$ is a boolean element iff X is an up-set and a down-set. In particular, each component is a boolean element.

Let $\Pi_0(P)$ be the quotient of P by the equivalence relation generated by the relation of comparability and $\Pi : P \longrightarrow \Pi_0(P)$ the canonical map.

Proposition 23 *For a lattice A, let i be the inclusion $i : B(A) \hookrightarrow A$ and let Π be the canonical map from A^* onto $\Pi_0(A^*)$. The following statements are equivalent:*

(1) $Ker\ i^* = Ker\ \Pi$;

(2) the map $I_0 : (B(A))^{*+} \longrightarrow B(A^{*+})$ defined by the prescription $I_0(X) = \{r \in A^* : r \cap B(A) \in X\}$ is an isomorphism.

Proof.
Let $i : B(A) \hookrightarrow A$ and let $j : B(A^{*+}) \hookrightarrow A^{*+}$ be the canonical inclusions. Then the map i^{*+} factors through j: $i^{*+} = j \circ I_0$. From the proposition 4 we have that i^{*+} is a monomorphism and then the same is true for I_0.

Assume (2). Let $p, q \in A^*$ such that $i^*(p) = i^*(q)$, that is, $p \cap B(A) = q \cap B(A)$. Since I_0 is an epimorphism, there is some $X \in (B(A))^{*+}$ such that $I_0(X) = \Pi(p)$. So, $p \in I_0(X)$, which implies $q \cap B(A) \in X$, i.e. $q \in I_0(X)$. Therefore, $\Pi(p) = \Pi(q)$.

If $\Pi(p) = \Pi(q)$, then there is some n such that $p \bowtie_n q$. But $p \subseteq q$ implies that $p \cap B(A) = q \cap B(A)$ because prime filters in $B(A)$ are maximal filters. Reasoning by induction on n we conclude that $i^*(p) = i^*(q)$.

Conversely, it suffices to show that (1) implies that I_0 is an epimorphism. Let $\Pi(p)$ be a component of A^* and set $X = \{r \cap B(A) | r \in \Pi(p)\}$. Therefore, $I_0(X) = \{q : q \cap B(A) = p \cap B(A)\}$. From (1) we deduce $I_0(X) = \Pi(p)$. The result follows taking into account that every boolean element of A^{*+} is a union of components.□

Proposition 24 *If A is a Heyting algebra that satisfies the Stone condition, then condition (1) above holds.*

Proof.
Let $p, q \in A^*$ and suppose q maximal. Then, $p \subseteq q$ if and only if $p \cap B(A) = q \cap B(A)$. Indeed, if $x \in p$ then $\neg\neg x \in p \cap B(A)$. Since q maximal, $x \in q$ or $\neg x \in q$. Then, $x \in q$. The converse follows from the maximality of prime filters in $B(A)$.

It is a known fact that if A satisfies the Stone condition then every prime filter of A is contained in a unique maximal. From the argument in the preceeding paragraph we can see that if $p \cap B(A) = q \cap B(A)$ then p and q are contained in the same maximal filter. Hence, $\Pi(p) = \Pi(q)$.□

Lemma 25 *If $p \bowtie_n q$, then $p \in \Diamond^n[q)$.*

Proof.
By induction on n. For $n = 1$ we have two possibilities.

(1) There exists r such that $p \geq r,\ r \leq q$.

(2) There exists r such that $p \leq r,\ r \geq q$.

By a direct computation of $\Diamond^1[q)$ in both cases and using remark 22 it is immediate that $p \in \Diamond^1[q)$. Suppose the statement true for $k < n$ and let $p \bowtie_{k+1} q$. Therefore there is some r such that $p \bowtie_1 r$ and $r \bowtie_k q$. By the inductive hypothesis, $r \in \Diamond^k[q)$. Hence $[r) \subseteq \Diamond^k[q)$ which implies that $\Diamond^1[r) \subseteq \Diamond^1(\Diamond^k)[q) = \Diamond^{k+1}[q)$. From the proof for $n = 1$ we have that $p \in \Diamond^1[r)$. Therefore $p \in \Diamond^{k+1}[q)$.□

Both the possibility and necessity operators always exist in A^{*+}. For $X \in A^{*+}$, $\Diamond(X)$ (respectively $\Box(X)$) is the union of the components C of A^* such that $C \cap X \neq \emptyset$ (respectively such that $C \subseteq X$).

Lemma 26 *If A is a bi-Heyting-algebra such that for some n, A^* is n-linked, then:*

(1) the inclusion $i : B(A) \hookrightarrow A$ has adjoints $\diamondsuit \dashv i \dashv \Box$.

(2) the map e_A preserves $\diamondsuit$ and $\Box$.

Proof.
If $X \in A^{*+}$, then $\diamondsuit(X) = \bigcup_{C \cap X \neq \emptyset} C$. Let $p \in \diamondsuit(X)$. Therefore, there exists $q \in X$ such that $p \in \Pi(q)$. Hence, $p \bowtie_n q$, which implies by a previous lemma $p \in \diamondsuit^n[q)$. But $\diamondsuit^n[q) \subseteq \diamondsuit^n(X)$, so $p \in \diamondsuit^n(X)$, and thus the equality $\diamondsuit^n(X) = \diamondsuit(X)$ follows. By unicity of adjoint we have also $\Box^n(X) = \Box(X)$.

The map e_A preserves the operators $\neg$ and $\sim$, therefore: $e_A(\diamondsuit^n(x)) = e_A((\sim\neg)^n(x)) = (\sim\neg)^n(e_A(x)) = \diamondsuit(e_A(x))$. We have also: $e_A(\diamondsuit^{n+1}(x)) = \diamondsuit(e_A(x))$, because n-linked implies $n+1$-linked. Therefore, $\diamondsuit^n(x) = \diamondsuit^{n+1}(x)$, since e_A is a monomorphism. But this equality implies: $\diamondsuit(x) = \diamondsuit^n(x)$. Therefore, the inclusion has a left adjoint $\diamondsuit$ wich is preserved by e_A. We can deduce in a dual way the existence of the right adjoint $\Box$, which is also preserved by e_A.

Corollary 27 *If A is a DH-algebra and A^* is n-linked, then the operators $\Longrightarrow$ and $\backslash\backslash$ exist on A and are preserved by the map e_A. In particular, the unary operators α and β defined by:*

$$\alpha(x) = x' \backslash\backslash x,$$

$$\beta(x) = x' \Longrightarrow x$$

exist and are preserved by e_A.

Proof.
It follows from lemma 21 and lemma 26.$\Box$.

2.2 n-valued Łukasiewicz algebras

An algebra $< A, \vee, \wedge, ', s_1, ..., s_{n-1}, 0, 1 >$ is an *n-valued Łukasiewicz algebra* if $< A, \vee, \wedge, ', 0, 1 >$ is a De Morgan algebra and the unary operators s_i,

$i = 1, ..., n-1$ satisfy:

$$\begin{cases} (L1)\ s_i(x \vee y) = s_i x \vee s_i y \\ (L2)\ s_i s_j x = s_j x, j = 1, .., n-1 \\ (L3)\ s_i(x') = (s_{n-i}x)' \\ (L4)\ s_i x \vee (s_i x)' = 1 \\ (L5)\ s_1 x \leq \ldots \leq s_{n-1} x \\ (L6)\ If (s_i x = s_i y) for i = 1, ..., n-1, then x = y. \end{cases}$$

For an equational definition, see [3].

Remark 28 (1) If A is a 5-valued Łukasiewicz algebra then $\Box$, α, β, $\Diamond$ are the operators s_i, for $i = 1, ..., 4$ (see [4]).

(2) The operators $s_1, ..., s_{n-1}$ of an n-valued Łukasiewicz algebra have no intrinsic definition for $n > 5$. Nevertheless, the fact that e_A preserves $s_1, ..., s_{n-1}$ is true for every n, as we shall see in this section.

(3) It is well known that if A is an n-valued Łukasiewicz algebra then A is a Heyting algebra ([7]) and the Stone condition holds in A. Therefore, according to 23 and 24 $B(A^{*+}) \approx (B(A))^{*+}$. From now on in this section we identify $B(A^{*+})$ with $(B(A))^{*+}$.

(4) In n-valued Łukasiewicz algebras every component $\Pi(p)$ of A^* is a chain of at most n elements and $\Pi(p) = \Pi(g_A(p))$, where g_A is the Birula-Rasiowa transformation. In fact, every component $\Pi(p)$ is determined by the maximal filter $r = p \cap B(A)$ of $B(A)$ because $q \in \Pi(p)$ iff $q \cap B(A) = r$. Moreover, $\Pi(p) = \{{s_1}^{-1}(r), {s_2}^{-1}(r), \ldots, {s_{n-1}}^{-1}(r)\}$.

Following the terminology of Fidel ([5]), for $i = 1, ..., n-1$ we let g_i be the functions s_i^*, i.e., for $p \in A^*$ define $g_i(p) = s_i^{-1}(p)$.

The operators g_i, $i = 1, ..., n-1$ have the following properties ([5]):

$$\begin{cases} (K3)\ p \subseteq q \text{ implies } g_i(p) \subseteq g_i(q) \text{ for } i = 1, ..., n-1 \\ (K4)\ g_i \circ g_j = g_i \text{ for } i, j = 1, ..., n-1 \\ (K5)\ g_A \circ g_i = g_{n-i} \\ (K6)\ A^* = \bigcup_{i=1}^{n-1} g_i(A^*) \\ (K7)\ g_i \leq g_{i+1} \text{ for } i = 1, ..., n-2 \end{cases}$$

For every $X \in A^{*+}$ set: $\Sigma_i(X) = s_i^{*+}(X)$, i.e., $\Sigma_i(X) = \{p \in A^* : g_i(p) \in X\}$.

We have that $e_A(s_i(x)) = \Sigma_i(e_A(x))$ for $i = 1, ..., n-1$, since e is a natural transformation.

Proposition 29 *The algebra* $< A^{*+}, \cup, \cap, n, \Sigma_1, ..., \Sigma_{n-1}, 0, 1 >$ *is an n-valued Łukasiewicz algebra.*

Proof.
From (L1) and (L3) we deduce that s_i is a lattice morphism, for $i = 1, ..., n-1$. Therefore, conditions (L1) and (L2) hold for Σ_i, $i = 1, ..., n-1$, because $(\ )^{*+}$ is a functor. (L3) follows from (K5).

To prove (L4) , we observe first that $\Sigma_i(e_A(x)) \cup n(\Sigma_i(e_A(x)) = 1$, which follows from the equalities $e_A(s_i(x)) = \Sigma_i(e_A(x))$ and $e_A(s_i(x)') = n(\Sigma_i(e_A(x)))$. Let $X \in A^{*+}$. It is well known that $X = \bigcup_{p\in X}(\bigcap_{x\in p} e_A(x))$. Therefore $\Sigma_i X = \bigcup_{p\in X}(\bigcap_{x\in p} \Sigma_i(e_A(x))$. Hence, $\Sigma_i X$ is boolean (increasing and decreasing). It is easy to see that $\Sigma_i X = g_A(\Sigma_i X)$, and this implies that $(\Sigma_i X)^c = n(\Sigma_i X)$.

Condition (L5) follows from (L5) for the operators s_i and the fact that $(\ )^{*+}$ is a functor.

The Moisil determination principle (L6) is proved considering the form of the components $\Pi(p)$. $\square$

Corollary 30 *The double dual* $(\ )^{*+}$ *is an endofunctor on the category* $\mathcal{L}_n$ *of n-valued Łukasiewicz algebras and e is a natural transformation such that for every object A of* $\mathcal{L}_n$ e_A *is conservative.*

2.3 Nelson algebras

A De Morgan algebra is a *Kleene algebra* if it satisfies the following additional condition

$$(K)\ \ \forall x, y\ (x \wedge x' \leq y \vee y')$$

A *quasi-Nelson algebra* is a Kleene algebra A such that for each pair $x, y \in A$, the implication $x \rightarrow (x' \vee y) = x \rhd y$ exists. The binary operation $\rhd$ so defined is called *weak implication*.

A *Nelson algebra* is a quasi-Nelson algebra satsifying the equation:

$$(N)\ (x \wedge y) \rhd z = x \rhd (y \rhd z).$$

(These algebras are called *N-lattices or quasi-pseudo-Boolean algebras* by H. Rasiowa).

The class of Nelson algebras constitutes a variety in the sense that it may be defined by a set of equations (see [1]).

Lemma 31 *If A is a Kleene algebra then so is A^{*+}.*

Proof.
Let $X, Y \in A^{*+}$, i.e., increasing subsets of A^*. We must prove that $X \cap n(X) \subseteq Y \cup n(Y)$. Since by definition $n(X) = (g(X))^c$, we are reduced to prove that $X \cap (g(X))^c \subseteq Y \cup (g(Y))^c$ or, what amounts to the same using the properties of set-theoretical complement, that

$$X \cap g(Y) \subseteq Y \cup g(X)$$

Notice first that since g is an anti-isomorphism of period 2, $r \in g(X)$ iff $g(r) \in X$.

Let $r \in X \cap g(Y)$. By the above remark, $r \in X$ and $g(r) \in Y$. Since A is a Kleene algebra, either $r \subseteq g(r)$ or $g(r) \subseteq r$. (See [3]). In the first case, $g(r) \in X$, i.e., $r \in g(X)$ (using the remark once more). In the second, $r \in Y$. In either case we have the conclusion.□

Because our previous results we have:

Corollary 32 *If A is a quasi-Nelson algebra, so is A^{*+}.*

Theorem 33 (Cignoli, Monteiro) *A quasi-Nelson algebra A is a Nelson algebra if and only if it satisfies the interpolation property.*

Recall that a Kleene algebra satisfies the *interpolation property* iff for every pair r and s of prime filters such that

(1) $r \subseteq g(r)$

(2) $r \subseteq g(s)$

(3) $s \subseteq g(r)$

(4) $s \subseteq g(s)$

there exists a prime filter q such that $r \subseteq q$, $s \subseteq q$, $q \subseteq g(r)$ and $q \subseteq g(s)$. That is r, s and q satisfy $r \subseteq q \subseteq g(r)$ and $s \subseteq q \subseteq g(s)$. Notice that 2) and 3) are equivalent conditions.

The following is known (see [13]) but we give a proof for the sake of completeness.

Theorem 34 *If A is a Nelson algebra, then so is A^{*+}.*

Proof.
Given $X, Y, Z \in A^{*+}$, we have to check that

$$(X \cap Y) \rhd Z = X \rhd (Y \rhd Z)$$

where $X \rhd Y = X \rightarrow (g(X)^c \cup Y)$.

By spelling out the definition of $\rhd$ we have to check that

$$(X \cap Y) \rightarrow (g(X \cap Y)^c \cup Z) = X \rightarrow (g(X)^c \cup (Y \rightarrow (g(Y)^c \cup Z))).$$

But notice that we have the following equivalences for p to belong to the term on the left:

$$\frac{\dfrac{p \in (X \cap Y) \rightarrow (g(X \cap Y)^c \cup Z)}{\forall q \supseteq p \ (q \in X \cap Y \ \& \ q \in g(X \cap Y) \ \Rightarrow \ q \in Z)}}{\forall q \supseteq p \ (q, g(q) \in X \cap Y \ \Rightarrow \ q \in Z)}$$

The first equivalence follows from the definition of $\rightarrow$ and the last from the trivial fact that $r \in g(X)$ iff $g(r) \in X$.

Similarly, the following is an equivalent condition for p to belong to the term on the right:

$$\frac{\forall q \supseteq p \ (q \in X \ \Rightarrow \ q \in g(X)^c) \text{ or } (q \in Y \ \Rightarrow \ (g(Y)^c \cup Z))}{\forall q \supseteq p \ (q \in X \ \Rightarrow \ q \in g(X)^c) \text{ or } (\forall r \supseteq q \ (r \in Y \ \Rightarrow \ r \in g(Y)^c \cup Z))}$$

To finish the proof, we have to check the equivalence between

(1) $\forall q \supseteq p \ (q, g(q) \in X \cap Y \ \Rightarrow \ q \in Z)$

(2) $\forall q \supseteq p\ (q \in X \ \Rightarrow\ q \in g(X)^c)$ or $(\forall r \supseteq q\ (r \in Y \ \Rightarrow\ r \in g(Y)^c \cup Z))$

(2)→(1) : clear.

(1)→(2) : assume (1) and suppose that (2) is not true, i.e., there is some $q \supseteq p$ such that $q, g(q) \in X$ and some $r \supseteq q$ such that $r, g(r) \in Y$ but $r \notin Z$.

Since $r \supseteq q$, $g(r) \subseteq g(q)$. But Y is an upset and hence $g(q) \in Y$. Thus $g(q) \in X \cap Y$ and $q \in X$. Assume that $q \in Y$. From (1), $q \in Z$ and so $r \in Z$, a contradiction. Therefore $q \notin Y$ and $q \subseteq g(q)$.

Similarly, $r \in X \cap Y$ and $g(r) \in Y$ but $g(r) \notin X$. This clearly implies that $g(r) \subseteq r$.

By interpolation, there is some s such that $s \supseteq q, g(r)$. Thus $s \in X \cap Y$. But $g(s) \supseteq q, g(r)$. Hence $g(s) \in X \cap Y$ and, by (1), $s \in Z$. Since Z is an up-set, $r \in Z$, a contradiction.□

Corollary 35 *The double dual ( )$^{*+}$ is an endofunctor on the category $\mathcal{N}$ of Nelson algebras and e ia a natural transformation such that for every object A of $\mathcal{N}$ e_A is conservative.*

3 Completeness Theorems

In this section we show the relation between completeness theorems for some extensions of intuitionistic propositional calculus (briefly: IPC) and the embedding theorem for the corresponding distributive lattices with further structure. The basic result is one already studied:

Theorem 36 *The evaluation map $e_D : D \to D^{*+}$ is conservative and conditionally bi-Heyting*

3.1 Modal symmetric propositional calculus

Modal symmetric propositional calculus (briefly: MSPC) was introduced by Moisil in 1942 (see [11]). This calculus is an extension of the intuitionistic propositional calculus (IPC). In fact,

$$MSPC = IPC + DM + CR$$

where DM and CR stand for the following axioms and rule of inference, respectively:

$$(DM) \qquad \begin{cases} x \longrightarrow x'' \\ x'' \longrightarrow x \end{cases}$$

$$(CR) \qquad \frac{x \longrightarrow y}{y' \longrightarrow x'}$$

The algebraic counterpart of MSPC (via the Lindenbaum-Tarski algebra) is the structure of a DH-algebra (see 2.1).

Let $T = (L, \Sigma)$ be a theory, where L is a language of MSPC (i.e., a language whose logical operators are $\wedge, \vee, \top, \bot, \rightarrow, \neg$ and $(\ )'$) and Σ is a set of sentences (of that language). Let A be the Lindenbaum-Tarski algebra of T. This is clearly a DH-algebra. If σ is a formula, we let $[\sigma]$ be its equivalence class. We mention the following known result which can be proved syntactically:

Proposition 37 *Let $\sigma \in L$. Then $T \vdash \sigma$ iff $[\sigma] = 1$ in A.*

The appropriate notion of model for MSPC is that of a g-Kripke model. A *g-Kripke model* ([6]) is a pair (M, g) where $M = (P, \leq, K)$ is a Kripke model and g is an involution from P onto its dual .

The next theorem, implicit in [6], uses the embedding theorem 10 to prove adequacy of MSPC.

Theorem 38 *Let σ and $T = (L, \Sigma)$ be a formula and a theory, respectively, in a language L of MSPC. Then $T \vdash \sigma$ iff σ is valid in every g-model of T.*

Proof.
Let σ be valid in all g-Kripke models of T. Consider the triple $(A^*, \subseteq, K)$, where K is the composition of the canonical map $[\ ]$ and the evaluation map e_A. It is clear that $(A^*, \subseteq, K, g)$ is a g-Kripke model of T, and both $[\ ]$ and e_A are morphisms in $\mathcal{DH}$. Therefore, $K(\sigma) = 1$. Since e_A is conservative, $[\sigma] = 1$ in A and this implies that $T \vdash \sigma$ is a theorem, by proposition 37.□

3.2 n-valued Łukasiewicz propositional calculus

An *n-valued Łukasiewicz algebra* can be characterized (see [7]) as an algebra $< A, \vee, \wedge, \longrightarrow, (\ )', s_1, s_2, .., s_{n-1}, 0, 1 >$ such that $< A, \vee, \wedge, \longrightarrow, (\ )', 0, 1 >$

is a DH-algebra and the unary operations s_i, $i = 1, 2, .., n-1$ satisfy:

$$\begin{cases} (L1), (L2), (L3), (L4) \text{ of } 2.2 \\ (L5') \quad s_1 x \vee x = x \\ (L6') \quad s_i(x \longrightarrow y) = \bigwedge_{j=i}^{n-1} (s_j x \longrightarrow s_j y). \end{cases}$$

This algebraic characterization allows us to define n-valued Łukasiewicz propositional calculus (nLPC) as an extension of IPC by means of the De Morgan negation $(\ )'$, the modal operators s_i for $i = 1, .., n-1$ and the deduction rules of substitution, modus ponens and the Gödel rule:

$$\frac{x}{s_1(x)}$$

The added logical axioms are those corresponding to the algebraic ones.

We can show the relationship between the theorem of double dual functor in the case of category $\mathbf{n}\mathcal{L}$ and adequacy for nLPC in the same way that we did for MSPC.

This gives a proof of the following (unpublished) result of [5]

Theorem 39 (M. Fidel) *Let σ and $T = (L, \Sigma)$ be a formula and a theory, respectively, in a language L of nLPC. Then $T \vdash \sigma$ iff σ is valid in every nL-model of T.*

4 Conservative extensions

As a final application of our methods, we shall prove conservativeness results among some extensions of coherent logic.

The first one asserts the following theorem

Theorem 40 (C. Rauszer) *Bi-intuitionistic propositional calculus (BIPC) is a conservative extension of intuitionistic propositional calculus (IPC)*

Before we go into the proof, let us be more precise about the formulation of BIPC. As logical connectives we take $\wedge, \top, \vee, \bot, \rightarrow, \backslash$. Notice that we can define $\neg p = p{\rightarrow}\bot$ and $\sim p = \top \setminus p$. To formulate BIPC we add to the formulation of IPC in [9] the following two rules of inference:

$$\frac{p \setminus q \vdash r}{p \vdash q \vee r} \quad \frac{p \vdash q \vee r}{p \setminus q \vdash r}$$

The theorem may now be stated more precisely as follows

Theorem 41 *Assume that σ and $T = (L, \Sigma)$ are a formula and a theory, respectively, in a language L of IPC. Then*

$$T \vdash_{BIPC} \sigma \;\; \textit{iff} \;\; T \vdash_{IPC} \sigma$$

Proof.
If Var is the set of propositional variables of L, we let $L^{\sim}$ be the language genereted by Var using the logical operators from BIPC and we let H be the Lindenbaum-Tarski algebra of T relative to IPC. The composite $e_H \circ [\,]$ may be extended to a map $v : L^{\sim} \longrightarrow H^{*+}$ by recursion. In fact, if p is a variable, then $v(p) = e_H \circ [p]$. The definition for the other clauses is obvious. For $\setminus$, we define

$$v(p \setminus q) = v(p) \setminus v(q)$$

Notice that this makes sense, since H^{*+} is a bi-Heyting algebra.

To conclude the proof, assume that $T \vdash_{BIPC} \sigma$. Then σ is valid in the Kripke model $(H^{*+}, \subseteq, v)$, i.e., $v(\sigma) = 1$ and this implies that $T \vdash_{IPC} \sigma$, since e_H is conservative.$\Box$

There is a categorical formulation of conservative results in [10]. To formulate the particular case under consideration, let $\mathcal{H}$ and $\mathcal{BH}$ be the categories of of Heyting algebras and bi-Heyting algebras, respectively. Assume that $U : \mathcal{BH} \longrightarrow \mathcal{H}$ is the obvious forgetful functor and let $F : \mathcal{H} \longrightarrow \mathcal{BH}$ be its left adjoint. Thus, UFH is the free bi-Heyting algebra generated by H

Theorem 42 *The unit $Id \stackrel{\eta}{\longrightarrow} UF$ of the adjunction $F \dashv U$ is a monomorphism*

Proof.
The bi-Heyting algebra $U(F(H))$ is freely generated by H. Therefore, given the evaluation map e_H, whose codomain H^{*+} is a bi-Heyting algebra, we have a biHeyting map f such that the following diagram commutes:

$$\begin{array}{ccc} & \eta_H & \\ H & \longrightarrow & U(F(H)) \\ e_H \downarrow & \swarrow_f & \\ H^{*+} & & \end{array}$$

Hence, η_H is a monomorphism, because e_H is a monomorphism.□

Remark 43 Lindenbaum-Tarski algebras give a way of constructing the free bi-Heyting algebra over a given Heyting algebra H. Indeed, present H as the Lindenbaum-Tarski algebra of a theory $T = (L, \Sigma)$ in IPC. Then the free bi-Heyting algebra over H is the Lindenbaum-Tarski algebra of $T^{\sim} = (L^{\sim}, \Sigma)$, where $L^{\sim}$ is as above (the language generated by the set of propositional variables Var obtained by using the logical operators of BIPC).

Similarly it is possible to see that the constructive calculus with strong negation (i.e., the logical counterpart of Nelson algebras) has a conservative extension NMSPC. This calculus has the language whose logical operators are $\wedge, \vee, \rightarrow, (\ )'$ and a new one $\rhd$ which is defined by

$$x \rhd y = x \rightarrow (x' \vee y)$$

The axioms and rules are those of MSPC with three further axioms

$$\begin{array}{ll} (K) & (x \wedge x') \rightarrow (y \vee y') \\ (N) & \begin{cases} ((x \wedge y) \rhd z) \rightarrow (x \rhd (y \rhd z)) \\ ((x \rhd (y \rhd z) \rightarrow ((x \wedge y) \rhd z) \end{cases} \end{array}$$

and one new rule of inference

$$(MNP) \qquad \frac{x \rhd y \qquad x}{y}$$

References

[1] Brignole,D. and A. Monteiro. [1967] Caractérisation des algèbres de Nelson par égalités. I, II Proc. Japan Acad. 43. 279-285.

[2] Cignoli,R. [1986] The class of Kleene algebras satisfying an interpolation property and Nelson algebras. Algebra Universalis 23. 262-292.

[3] Cignoli,R. [1970] Moisil algebras. Notas de Lógica Matemática N^0 27. Instituto de Matem. UN del Sur, B. Blanca.

[4] Cignoli,R. and M.Sagastume de Gallego. [1983] Dualities for some De Morgan algebras with operators and Lukasiewicz algebras. J. Austral. Math. Soc. (Series A) 34. 377-393.

[5] Fidel,M. [unpublished] Un cálculo modal correspondiente a las álgebras de Moisil de orden n.

[6] Galli,A. and M.Sagastume. [to appear] Symmetric-intuitionistic connectives. Proceedings of the X Latinamerican Simposium on Mathematical Logic.

[7] Iturrioz,L. [1977] Łukasiewicz and symmetric Heyting algebras. Z.Math.Logik Grundlagen Math. 23. 131-136.

[8] Johnstone,P.T. [1982] *Stone Spaces*. Cambridge University Press.

[9] Lambek,J. and P.J.Scott. [1986] *Introduction to higher order categorical logic*. Cambridge University Press.

[10] Makkai,M. and G.E. Reyes. [1995] Completeness results for intuitionistic and modal logic in a categorical setting. Annals of Pure and Applied Logic 72. 25-101.

[11] Monteiro,A. [1980] Sur les algébres de Heyting symétriques. Portugaliae Mathematica. Vol.39. 1-237.

[12] Reyes,G.E. and H.Zolfaghari. [1996] Bi-Heyting algebras, toposes and modalities. Journal of Philosophical Logic 25. 25-43.

[13] Vakarelov,D. [1977] Notes on N-lattices and constructive logic with strong negation. Studia Logica 36. 109-125.