

Generic figures and their glueings

A constructive approach to functor categories

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Dedicated to the memory of John Macnamara
1929 – 1996

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Introduction

Although there are several textbooks on topos theory, we feel that ours fills a definite need.

The fully fledged notion of a Grothendieck topos seems formidable to the beginner. Too many notions are invoked in its definition and most of the examples presuppose a good deal of mathematical experience. This makes the subject difficult both for beginner mathematicians as well as for people like logicians, linguists and psychologists who would like to know what topos theory is about and how to use it as a tool in their work.

The usual alternative is to start with the axiomatic approach to topos theory, delving into elementary toposes from the beginning. Although a lot of theory can be learned in this way, this approach has the disadvantage that one doesn't have the main examples of which elementary topos are an axiomatization of. In fact, we feel that logically as well as psychologically, well chosen examples are needed to understand an axiomatization. But elementary toposes are an axiomatization of Grothendieck toposes and we are back to the previous difficulty!

A few years ago, Lawvere suggested another alternative: to introduce topos theory through presheaf toposes or, equivalently, \mathbb{C} -sets. These are categories whose objects result from the glueing of simpler ones, the generic figures. These categories are Grothendieck toposes which do not involve the notion of a Grothendieck topology, making them much easier to understand and work with them.

This approach has several advantages, the most important being the simplicity of the categories involved and the rich theory to which they give rise. Several phenomena which distinguish toposes from the ordinary category of sets appear already at this level. In fact, some of these toposes have played an important role in the search of further axioms on a topos to define toposes of space, toposes of motion, etc.

Although Lawvere and Schanuel developed this approach in their beautiful book ([24]), the scope of that work did not allow them to go into a systematic study of presheaf toposes and their connections.

Our book follows this alternative to its bitter end. After the definition of a category of \mathbb{C} -sets we consider six easy to understand examples which,

furthermore, have clear graphical representations: categories of sets, bisets, bouquets, graphs, reflexive graphs and evolutive sets. They keep us company throughout the whole book to illuminate new material, interpret general results and suggest new theorems.

The description of our aims implicitly defines the reader that we have in mind: a beginner mathematician or scientist or philosopher (of an arbitrary age) who would like to take advantage of the rich structure and theory that presheaf toposes have to prepare himself or herself either for further study or for applications of the theory described.

We mentioned the book by Lawvere and Schanuel. This is an excellent work to learn the basic notions of category theory with well chosen examples and clear motivations. It is indeed a first introduction to categories. Our book presupposes the subject covered by theirs and prepares the reader to study more advanced works like the book by MacLane and Moerdijk (see [26]), a very readable account of topos theory that, however, can hardly be described as ‘A first introduction’ as the authors advertise it.

After describing the ‘topos’ of this book in the literature, a few words are in order to motivate \mathbb{C} -sets.

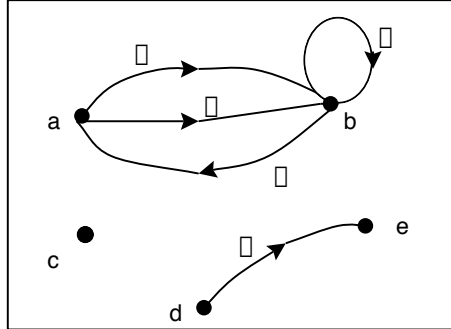
Assume that we want to describe and study a class of complicated structure, say graphs. Of course, a graph G is a collection of vertices and arrows with some relations of incidence between them. Usually, a graph is formalized as a couple of sets G_1 (the arrows) and G_0 (the vertices) together with two maps $s : G_1 \rightarrow G_0$ (source) and $t : G_1 \rightarrow G_0$ (target). These maps associate to an arrow its source and its target, respectively.

There is an alternatively, and we hope to show, more illuminating way of considering a graph, namely as consisting of figures of two shapes; the V -figures or figures of shape V (‘vertex’) and A -figures or figures of shape A (‘arrow’) subject to some relations that we describe in terms of ‘change of figures’. Looking at the arrow itself as a graph, we see that it has exactly one A -figure (the arrow) and two V -figures (the source and the target). In a similar vein, we may look at the vertex as a graph with one vertex and no arrows. Thus, these figures that we called ‘generic’ constitute a category

$$V \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} A$$

(with identities omitted). Each morphism is called a ‘change of figure’. For instance $s : V \rightarrow A$ allows to change an A -figure into a V -figure, namely into the V -figure which is the source of the original A -figure.

Incidence relations may be formulated in terms of right actions. To make this discussion more explicit, consider the following graph



We have 5 A -figures (or arrows): $\alpha, \beta, \gamma, \epsilon$ and 5 V -figures (or vertices): a, b, c, d, e with the following table of right actions:

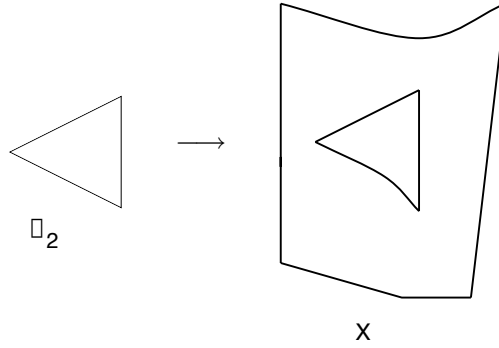
$$\begin{aligned} \alpha.s &= \beta.s = \gamma.t = a \\ \alpha.t &= \beta.t = \gamma.s = \delta.s = \delta.t = b \\ \epsilon.s &= d \\ \epsilon.t &= e \end{aligned}$$

The incidence relations can be read at once from the table. Notice, furthermore that in an obvious sense, the whole graph may be obtained by glueing its generic figures.

A morphism of graphs is a function that sends vertices into vertices, arrows into arrows and preserves the incidence relation, or what amounts to the same, the right action.

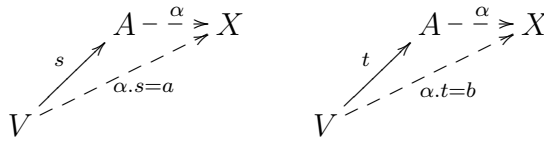
This way of analyzing complex mathematical structure in terms of simpler ones (generic figures) with a right action of change of figures has been implicit for a long time in mathematical practice, but it is only in our century that it was made explicit. In the so-called singular theory in Algebraic Topology, for instance, a topological space is viewed as consisting of figures ‘points’, ‘intervals’, ‘triangles’ and so on with changes of figures given by ‘extracting the end points of an interval’ or ‘extracting the sides of a triangle’ and so

on. In this case, the generic figures may be objectivized as actual continuous maps from the corresponding euclidean spaces into the space:



In a figurative way of talking, such a map is an ‘extraction’ of the euclidean figure from the space. The functors of singular homology and singular cohomology may be defined in this context.

To keep this intuition of ‘extraction of generic figures’, we shall use an alternative notation for figures and actions, two examples of which are given for the graph:



Later on we shall see how, thanks to Yoneda lemma, these dotted arrows may be interpreted as real morphisms, making the analogy with singular theory very close.

A few words about the contents: the book is divided into 14 chapters. The first five deal with categories of \mathbb{C} -sets. We study operations on \mathbb{C} -sets and the object Ω of truth values. After two chapters on adjointness in posets and in categories, we return to \mathbb{C} -sets to deal with logical operations definable in them. This chapter is followed by one on doctrines, i.e., categorical counterparts of classical and non-classical logics. The next three chapters deal with geometric morphisms which are the objective way to describe connections between categories of \mathbb{C} -sets. The last chapter deals with points of categories of \mathbb{C} -sets and their connection with models of some infinitary theories described in terms of the category of figures.

There are exercises in every chapter. They are important and several are presupposed later in the main text. Some chapters have ‘windows’. These are designed for the reader to have a glimpse on other territories not covered in the main text. Some are mainly bibliographical, others frankly philosophical.

This book originated in a two-year course (1991-1993) that Hooman Zolfaghari taught (from the point of view of generic figures and their glueings) for the benefit of a multidisciplinary group working on logical foundations of cognition. Besides the authors of this book, the group included John Macnamara. We felt that all of us should have a working knowledge of the tools that we were using in our work, rather than counting on the usual division of labour. This was a challenge since Macnamara, a psychologist at McGill, had never studied mathematics before. The result was very encouraging. We will never forget his joy in realizing that he could compute geometric morphisms between the main examples. As he put it, he was promoted from a christian to a lion.

During Reyes sabbatical year (1996/97) from Montreal University, the first two authors had an opportunity to try the notes of Zolfaghari’s course taken by the first author both in La Plata and in Santiago. The enthusiastic reception of the audience in La Plata launched us in the actual writing of this book.

It is a pleasure to thank people who have helped us during the long time of gestation.

First, our thanks go to Adriana Galli and Marta Sagastume for inviting Reyes to La Plata during the month of June 1996. Many thanks also to Renato Lewin who was responsible for the invitation of Reyes to the Pontificia Universidad Catolica de Santiago and for organizing a one year course on generic figures and their glueings at that university. The notes that the first author took in La Plata and in Santiago were the first version of this book. Among the participants to both courses we would like to mention, besides the organizers, Matias Menni, Hector Gramaglia, Gaston Argeri, Guillermo Ortiz and Carlos Martinez who, through many questions and remarks, forced us to better explain, change and improve the presentation in a number of places.

Last, but not least, we are in great debt to Bill Lawvere. We have freely used materials from his papers, books, talks and conversations. Furthermore his encouragement has meant a lot to us.

We dedicate this book to the memory of our friend John Macnamara.

The first author is grateful for the support from a grant of the Fonds du Québec pour la formation de chercheurs et l'aide à la recherche (FCAR) given to Macnamara and Reyes. The second author would like to thank the following institutions that made the actual writing possible: the Université de Montréal, for granting him a sabbatical leave, the Universidad de La Plata, for inviting him on two occasions as Visiting Professor, La Fundación Andes de Chile for helping him with financial support, the Pontificia Universidad Católica de Chile for appointing him Visitor Professor and offering him its facilities during his sabbatical leave and Canada's National Science and Engineering Research Council (NSERC) for maintaining its financial support during the time in which this book was written.

Montréal, spring 1998

1 The category of \mathbb{C} -Sets

Let \mathbb{C} be a category whose objects are thought of as generic figures and whose morphisms are thought of as changes of figures. A \mathbb{C} -Set is a family $X = (X(F))_{F \in \text{Ob}(\mathbb{C})}$ of sets $X(F)$ indexed by the objects of \mathbb{C} , which we call *F-figures of X*, together with a right action:

$$(\sigma \in X(F), F' \xrightarrow{f} F \in \mathbb{C}) \mapsto \sigma.f \in X(F')$$

$F - \overset{\sigma}{\succ} X$ which satisfies

$$\sigma.1_F = \sigma$$

and

$$(\sigma.f).g = \sigma.(f \circ g)$$

Notation: We will use the following notations for an F -figure of X ‘ $F - \overset{\sigma}{\succ} X$ ’ which stresses the aspect ‘extraction’ of a figure of X or ‘ $\sigma \in_F X$ ’ which makes clear the intuition that σ belongs to the container at the level F . We will also use the following diagram for the action of an arrow on a F -figure:

$$\begin{array}{ccc} & & F - \overset{\sigma}{\succ} X \\ & \nearrow f & \\ F' & \dashrightarrow \sigma.f & \end{array}$$

A *morphism* $X \xrightarrow{\Phi} Y$ of \mathbb{C} -Sets is a rule which sends F -figures of X into F -figures of Y and which is compatible with the change of figures, i.e., such that $\Phi(\sigma.f) = \Phi(\sigma).f$. In other words ‘a change of figure followed by a change of container ($\Phi(\sigma.f)$) is the same as a change of container followed by a change of figure ($\Phi(\sigma).f$)’. The following diagram will often be used in this context

$$\begin{array}{ccc} & & F - \overset{\sigma}{\succ} X \xrightarrow{\Phi} Y \\ & \nearrow f & \\ F' & \dashrightarrow \sigma.f & \end{array}$$

A more standard way of saying the same is that a \mathbb{C} -Set is a functor from the dual of \mathbb{C}

$$X : \mathbb{C}^{op} \longrightarrow Sets$$

and a morphism of \mathbb{C} -Sets is a natural transformation, i.e., Φ is a family of functions $\Phi = (\Phi_F)_{F \in Ob(\mathbb{C})}$ such that $\forall F' \xrightarrow{f} F \in \mathbb{C}$ the following diagram is commutative

$$\begin{array}{ccc} X(F) & \xrightarrow{\Phi_F} & Y(F) \\ (\cdot).f \downarrow & & \downarrow (\cdot).f \\ X(F') & \xrightarrow{\Phi_{F'}} & Y(F') \end{array}$$

namely that $\Phi_F(\sigma).f = \Phi_{F'}(\sigma.f)$. We let $Sets^{\mathbb{C}^{op}}$ be the category of contravariant functors and natural transformations, i.e., the category of \mathbb{C} -sets. Such a functor is called a *presheaf* on \mathbb{C} and, accordingly, $Sets^{\mathbb{C}^{op}}$ is called the *category of presheaves* on \mathbb{C} .

1.1 Examples of \mathbb{C} -Sets

- *Sets*. We can represent an object graphically and as a container .

Graphically:

$$X = \boxed{\begin{array}{ccc} \bullet & & \bullet \\ \bullet & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ \bullet & & \end{array}}$$

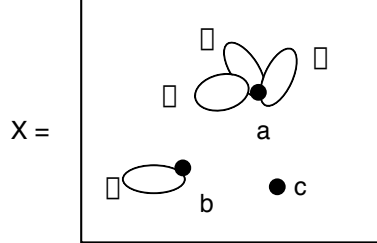
A morphism between two sets is a function that sends points into points.

As a container: take $\mathbb{C} = \mathbb{1}$, the category containing only one generic figure P (for *point*), with the trivial change of figure 1_P . A set X may be identified with the $\mathbb{1}$ -set or container whose P -figures (or points) are the

elements of $X : \frac{P - \sigma \triangleright X}{\sigma \in X}$ with the (trivial) action

$$\begin{array}{ccc} & P - \sigma \triangleright X & \\ 1_P \nearrow & & \dashrightarrow \\ P & \dashrightarrow & \sigma.1_P = \sigma \end{array}$$

Graphically:



represents the bouquet $X_1 \xrightarrow{u} X_0$ where $X_1 = \{\alpha, \beta, \gamma, \delta\}$, $X_0 = \{a, b, c\}$ and $u(\alpha) = u(\beta) = u(\gamma) = a$, $u(\delta) = b$. For obvious reasons we call these equations *relations of incidence*. A morphism between two such objects is a function (or rule) that sends loops into loops, vertices into vertices and respects the actions or, as we will often say, preserves the incidence relations. Thus if $f = (f_1, f_0)$ is such a rule from $X_1 \xrightarrow{u} X_0$ into $Y_1 \xrightarrow{v} Y_0$, α is sent into a loop $f_1(\alpha)$ whose vertex is $f_0(a)$.

As a container: take $\mathbb{C} = V \xrightarrow{v} L$, the category having two generic figures: V (for *vertex*), L (for *loop*) and one morphism v (for *extraction of the vertex* of the loop). The identities $1_V, 1_L$ were omitted from the above representation of \mathbb{C} . The bouquet X may be identified with the $(V \xrightarrow{v} L)$ -set whose V -figures are a, b, c and whose L -figures are $\alpha, \beta, \gamma, \delta$:

$$\frac{V \dashrightarrow X}{a, b, c} \quad \frac{L \dashrightarrow X}{\alpha, \beta, \gamma, \delta}$$

with obvious action. For instance

$$\begin{array}{ccc} & L \dashrightarrow X & \\ & \nearrow v & \\ V & \dashrightarrow & \end{array} \quad \begin{array}{ccc} & L \dashrightarrow X & \\ & \nearrow \alpha & \\ & \dashrightarrow \alpha.v=a & \end{array}$$

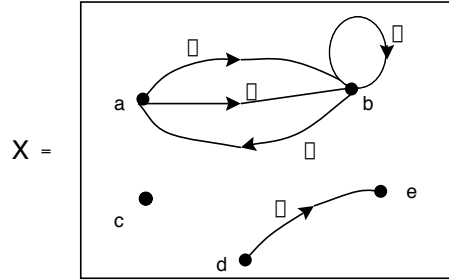
expresses the fact that the extraction of the vertex of α gives a . A morphism of $(V \xrightarrow{v} L)$ -sets is a rule that sends V -figures into V -figures and L -figures into L -figures in such a way that the incidence relations are preserved.

• *Graphs*. Objects are oriented multi-graphs and morphisms are graph homomorphisms. In other words an object X consists of two sets X_1, X_0 and

two functions $X_1 \begin{matrix} \xrightarrow{u_0} \\ \xrightarrow{u_1} \end{matrix} X_0$. The elements of X_1 are called *arrows*, those of X_0 *vertices*, $u_0(\alpha)$ the *source* of α , $u_1(\alpha)$ the *target* of α . A morphism $X \xrightarrow{f} Y$ is a couple of functions $X_1 \xrightarrow{f_1} Y_1$, $X_0 \xrightarrow{f_0} Y_0$ such that the following diagrams commute, namely, $f_0 u_0 = v_0 f_1$, $f_0 u_1 = v_1 f_1$:

$$\begin{array}{ccc} X_1 & \begin{matrix} \xrightarrow{u_0} \\ \xrightarrow{u_1} \end{matrix} & X_0 \\ f_1 \downarrow & & \downarrow f_0 \\ Y_1 & \begin{matrix} \xrightarrow{v_0} \\ \xrightarrow{v_1} \end{matrix} & Y_0 \end{array}$$

Graphically:



represents the graph $X_1 \begin{matrix} \xrightarrow{u_0} \\ \xrightarrow{u_1} \end{matrix} X_0$ where $X_1 = \{\alpha, \beta, \gamma, \delta, \epsilon\}$, $X_0 = \{a, b, c, d, e\}$ and $u_0(\alpha) = u_0(\beta) = a$, $u_0(\delta) = u_0(\gamma) = b$, etc., $u_1(\alpha) = u_1(\beta) = u_1(\delta) = b$, etc. A morphism between two such objects is a function (or rule) that sends arrows into arrows and vertices into vertices and preserves the incidence relations. Thus if $f = (f_1, f_0)$ is such a rule from $X_1 \begin{matrix} \xrightarrow{u_0} \\ \xrightarrow{u_1} \end{matrix} X_0$ into $Y_1 \begin{matrix} \xrightarrow{u_0} \\ \xrightarrow{u_1} \end{matrix} Y_0$, α is sent into an arrow $f_1(\alpha)$ whose source is $v_0(f_1(\alpha)) = f_0(a)$ and whose target is $v_1(f_1(\alpha)) = f_0(b)$.

As a container: take $\mathbb{C} = V \begin{matrix} \xrightarrow{s} \\ \xrightarrow{t} \end{matrix} A$, the category having two generic figures: V (for *vertex*), A (for *arrow*) and two morphisms s (for *extraction of the source* of the arrow) and t (for *extraction of the target* of the arrow). The identities 1_V , 1_A were omitted from the description of \mathbb{C} . The graph X may

be identified with the $(V \begin{smallmatrix} \xrightarrow{s} \\ \xrightarrow{t} \end{smallmatrix} A)$ -set whose V -figures are a, b, c, d, e and whose A -figures are $\alpha, \beta, \gamma, \delta, \epsilon$:

$$\frac{V \dashrightarrow X}{a, b, c, d, e} \quad \frac{A \dashrightarrow X}{\alpha, \beta, \gamma, \delta, \epsilon}$$

with obvious action. For instance

$$\begin{array}{ccc} & A \xrightarrow{\alpha} X & \\ & \nearrow \text{---} & \\ V & \xrightarrow{s} & \\ & \searrow \text{---} & \\ & \alpha.s=a & \end{array} \quad \begin{array}{ccc} & A \xrightarrow{\alpha} X & \\ & \nearrow \text{---} & \\ V & \xrightarrow{t} & \\ & \searrow \text{---} & \\ & \alpha.t=b & \end{array}$$

expresses the fact that the extraction of the source of α gives a and its target is b . A morphism of $(V \begin{smallmatrix} \xrightarrow{s} \\ \xrightarrow{t} \end{smallmatrix} A)$ -sets is a rule that sends V -figures into V -figures and A -figures into A -figures in such a way that the incidence relations are preserved.

• *Rgraphs*. Objects are oriented reflexive multi-graphs and morphisms are reflexive graph homomorphisms. In other words an object X consists of two sets X_1, X_0 and three functions

$$X_1 \begin{array}{c} \xrightarrow{u_0} \\ \xrightarrow{u_1} \\ \xleftarrow{i} \end{array} X_0$$

with $u_0i = u_1i = 1_{X_0}$. The elements of X_1 are called *arrows*, those of X_0 *vertices*, $u_0(\alpha)$ *source* of α , $u_1(\alpha)$ *target* of α and $i(a)$ *distinguished loop* whose source (and target) is a .

A morphism $X \xrightarrow{f} Y$ where Y is

$$Y_1 \begin{array}{c} \xrightarrow{v_0} \\ \xrightarrow{v_1} \\ \xleftarrow{j} \end{array} Y_0$$

is a couple of functions $X_1 \xrightarrow{f_1} Y_1, X_0 \xrightarrow{f_0} Y_0$ such that the following dia-

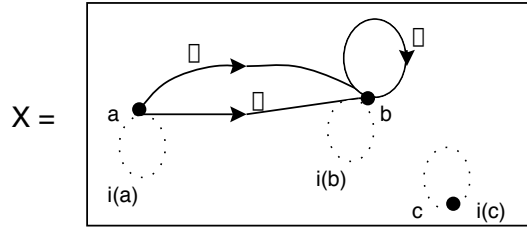
grams commute, namely, $f_0 u_0 = v_0 f_1$, $f_0 u_1 = v_1 f_1$ and $f_1 i = j f_0$.

$$\begin{array}{ccc} X_1 & \begin{array}{c} \xrightarrow{u_0} \\ \xrightarrow{u_1} \end{array} & X_0 \\ f_1 \downarrow & & \downarrow f_0 \\ Y_1 & \begin{array}{c} \xrightarrow{v_0} \\ \xrightarrow{v_1} \end{array} & Y_0 \end{array}$$

and

$$\begin{array}{ccc} X_1 & \xleftarrow{i} & X_0 \\ f_1 \downarrow & & \downarrow f_0 \\ Y_1 & \xleftarrow{j} & Y_0 \end{array}$$

Graphically:



represents the reflexive graph $X_1 \begin{array}{c} \xrightarrow{u_0} \\ \xrightarrow{u_1} \\ \xleftarrow{i} \end{array} X_0$ where

$$X_1 = \{\alpha, \beta, \gamma, i(a), i(b), i(c)\}, \quad X_0 = \{a, b, c\}$$

and $u_0(\alpha) = u_0(\beta) = a$, $u_0(\gamma) = b$, $u_1(\alpha) = u_1(\beta) = u_1(\gamma) = b$, $u_0 i(a) = u_1 i(a) = a$, etc. A morphism between two such objects is a function (or rule) that sends arrows into arrows, vertices into vertices, distinguished loops into distinguished loops and preserves the incidence relations. Thus if $f =$

(f_1, f_0) is such a rule from $X_1 \begin{array}{c} \xrightarrow{u_0} \\ \xrightarrow{u_1} \\ \xleftarrow{i} \end{array} X_0$ into $Y_1 \begin{array}{c} \xrightarrow{v_0} \\ \xrightarrow{v_1} \\ \xleftarrow{j} \end{array} Y_0$ $i(a)$ is sent into a distinguished loop $f_1(i(a))$ whose source (and target) is $f_0(a)$.

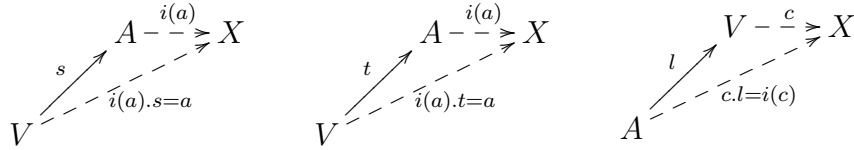
As a container: take $\mathbb{C} = V \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \\ \xleftarrow{l} \end{array} A \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}$ the category having two generic

figures: V (for *vertex*), A (for *arrow*) and morphisms s (for *extraction of the*

source of the arrow), t (for *extraction of the target* of the arrow) and l (for *extraction of the distinguished loop* from its vertex) such that $l \circ s = l \circ t = 1_V$, $\sigma = sol, \tau = tol$. Notice that we left identities out of the picture. The reflexive graph X may be identified with the $(V \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \\ \xleftarrow{l} \end{array} A \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array})$ -set whose V -figures are a, b, c and whose A -figures are $\alpha, \beta, \gamma, i(a), i(b)$ and $i(c)$:

$$\frac{V \dashrightarrow X}{a, b, c} \quad \frac{A \dashrightarrow X}{\alpha, \beta, \gamma, i(a), i(b), i(c)}$$

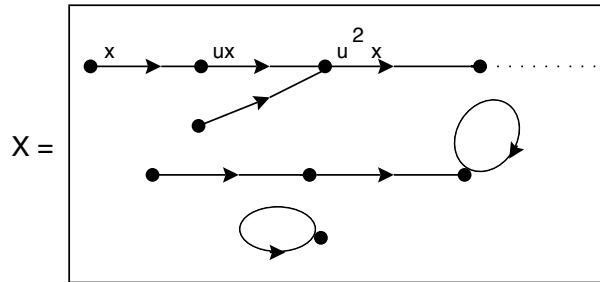
with obvious action. For instance the three diagrams



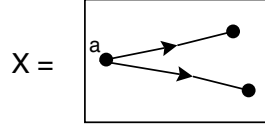
express three things: the extraction of the source of $i(a)$ gives a , the extraction of the target of $i(a)$ gives a and the extraction of the distinguished loop of c gives $i(c)$. A morphism of $(V \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \\ \xleftarrow{l} \end{array} A \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array})$ -sets is a rule that sends

V -figures into V -figures and A -figures into A -figures in such a way that the incidence relations are preserved.

- *Esets*. Objects are evolutive sets or deterministic, discrete dynamical systems and morphisms are functions that respect the evolution. An *evolutive set* X is a pair (X_0, u) where X_0 is a set and $X_0 \xrightarrow{u} X_0$ an ordinary endofunction. A morphism $X \xrightarrow{f} Y$, where $Y = (Y_0, v)$ is a function $X_0 \xrightarrow{f} Y_0$ such that $f(u(x)) = v(f(x))$ for all $x \in X$. We call the elements of X_0 the *elements (or states)* of X and u the *evolution* of X . Graphically, an evolutive set is represented as follows:



Note that

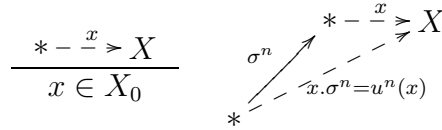


does not represent an evolutive set since the evolution of a should be uniquely determined.

As a container: we take as category of generic figures and changes of figures the category \mathbb{E} having one object $*$ and the iterations of a morphism



Thus, the morphisms are $1_*, \sigma, \sigma^2, \sigma^3, \dots$. In other words, \mathbb{E} is the free monoid on one generator (σ) seen as a category. An evolutive set $X = (X_0, u)$ may be identified with an \mathbb{E} -set whose elements are the elements of X_0 and whose action is $x.\sigma^n = u^n(x)$:



(The equation for the action $x.\sigma^n = u^n(x)$ is forced by definition of an action and the particular case $x.\sigma = u(x)$.) A morphism of \mathbb{E} -sets $X \xrightarrow{f} Y$ is a function $X_0 \xrightarrow{f} Y_0$ which preserves the evolution in the sense that $f(x.\sigma^n) = f(x).\sigma^n$, i.e., $f(u^n(x)) = v^n(f(x))$ where $X = (X_0, u)$ and $Y = (Y_0, v)$. Notice that f preserves the action if and only if $f(u(x)) = v(f(x))$.

1.2 Monomorphism, epimorphism, isomorphism

The notions of the title of this section are well-known and may be defined in any category. For \mathbb{C} -sets they may be described as follows:

Proposition 1.2.1 *Let $f : Y \rightarrow X$ be a morphism of \mathbb{C} -sets. Then*

(1) *f is a monomorphism iff $f_C : Y(C) \rightarrow X(C)$ is a (set-theoretical) injection, for every $C \in \mathbb{C}$*

(2) *f is an epimorphism iff $f_C : Y(C) \rightarrow X(C)$ is a (set-theoretical) surjection, for every $C \in \mathbb{C}$*

Remark 1.2.2 There is not a single, clear cut analogue of the notion of set-theoretical surjection in arbitrary categories. The ‘weakest’ analogue seems to be the notion of epimorphism. But other notions have been studied in the literature. Let us mention two: regular epimorphism and extremal epimorphism or surjections. Fortunately, all of these notions coincide in a category of presheaves (and, more generally in a topos).

(3) f is an isomorphism iff $f_C : Y(C) \rightarrow X(C)$ is a (set-theoretical) bijection, for every $C \in \mathbb{C}$

Proof.

We shall prove (1), leaving the other as exercises.

\rightarrow : Assume that $Y \xrightarrow{f} X$ is a monomorphism. Let $C \in \mathbb{C}$ and $a, b \in Y(C)$ such that $f_C(a) = f_C(b)$. By Yoneda, we may identify a, b with morphisms of \mathbb{C} -sets with domain C and codomain Y such that $f \circ a = f \circ b$. By definition of monomorphism, $a = b$.

\leftarrow : Let

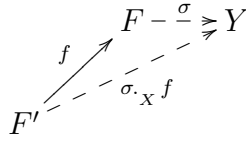
$$Z \begin{array}{c} \xrightarrow{u} \\ \xrightarrow{v} \end{array} Y \xrightarrow{f} X$$

be a diagram such that $f \circ u = f \circ v$. Then, for each $C \in \mathbb{C}$, $f_C u_C = f_C v_C$. Let $z \in Z$. Then $f_C(u_C(z)) = f_C(v_C(z))$. Since f_C is an injection, $u_C(z) = v_C(z)$. But z was arbitrary and so $u_C = v_C$ for each component, i.e., $u = v$. \square

In the category of sets there are canonical monomorphisms, namely inclusions, that ‘represent’ monomorphisms in the sense that for each monomorphism $f : Y \rightarrow X$ there is a unique subset $A \overset{i}{\hookrightarrow} X$ and a unique bijection $Y \xrightarrow{g} A$ such that $f = ig$.

Following the analogy with sets, we first define the notion of ‘sub-presheaf’ or ‘sub- \mathbb{C} -set’, which generalizes the notion of subset.

A *sub- \mathbb{C} -set* of a \mathbb{C} -set X is a \mathbb{C} -set Y with a morphism of \mathbb{C} -sets $i : Y \hookrightarrow X$ such that for each $F \in Ob(\mathbb{C})$, $i_F : Y(F) \subseteq X(F)$ is the set inclusion. We sometimes will find more convenient to use the following equivalent definition: a *sub- \mathbb{C} -set* Y of a \mathbb{C} -set X is a family $Y = (Y(F))_{F \in Ob(\mathbb{C})}$ closed under the action, namely, if σ is an F -figure of Y (and hence of X) and $F' \xrightarrow{f} F$ is a morphism of \mathbb{C} , then the result of the action in the sense of X , $\sigma \cdot_X f$ is an F' -figure of Y (and not only of X):



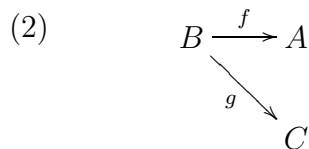
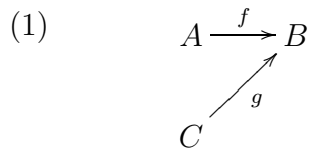
As an example, a subgraph of a graph is a collection of arrows and vertices such that whenever an arrow belongs to the collection, its source and target also belong to the collection.

EXERCISE 1.2.1

- (1) If $A \xrightarrow{i} X$ is a sub- \mathbb{C} set of X , then i is a monomorphism. Conversely, if $f : Y \rightarrow X$ is a monomorphism show the existence of a unique sub- \mathbb{C} set A of X and a unique isomorphism $Y \xrightarrow{g} A$ such that $f = ig$.
- (2) Let \mathbb{C} be a small category. Show that in the category of \mathbb{C} -sets the isomorphisms are precisely the monomorphisms which are also epimorphisms.
- (3) Let Δ_1 be the monoid of the endomorphisms of the arrow A of reflexive graphs. As a category, Δ_1 has one object and three arrows: 1 , δ_0 and δ_1 with the relations $\delta_i \delta_j = \delta_i$ (where δ_0 is the constant map that sends the whole arrow in its source and δ_1 the arrow in its target.) Show that $\text{Sets}^{\Delta_1^{op}}$ may be identified with the category of reflexive graphs.

EXERCISE 1.2.2

If \mathbb{C} is any of the categories below, interpret the category of \mathbb{C} -Sets both graphically and as a container:



$$(3) \quad A \xrightarrow{f} B \xrightarrow{g} C$$

$$(4) \quad V \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} A \xrightarrow{a} S$$

(5) A monoid with two generators σ and l

$$\begin{array}{c} \curvearrowright \square \\ * \curvearrowright \end{array}$$

such that $\sigma^n l = l$ for $n = 1, 2, 3, \dots$, and $ll = l$.

$$(6) \quad V \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} A \begin{array}{c} \xrightarrow{\delta_0} \\ \xrightarrow{\delta_1} \\ \xrightarrow{\delta_2} \end{array} T$$

where $\delta_0 t = \delta_1 s$, $\delta_1 t = \delta_2 s$ and $\delta_2 t = \delta_0 s$

$$(7) \quad V \begin{array}{c} \xrightarrow{v_0} \\ \xrightarrow{v_1} \\ \xrightarrow{v_2} \end{array} T$$

$$(8) \quad V \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} A \begin{array}{c} \xrightarrow{f_0} \\ \xrightarrow{f_1} \end{array} C$$

where $f_0 t = f_1 t$ and $f_0 s = f_1 s$.

2 Representable \mathbb{C} -sets and Yoneda lemma

Let \mathbb{C} be a category (whose objects and morphisms are thought of as generic figures and changes of figures). Every generic figure F may be considered itself as a container of generic figures or \mathbb{C} -set h_F whose F' -figures are the morphisms $F' \rightarrow F \in \mathbb{C}$ with the operation of composing to the right as action:

$$\frac{F' \dashrightarrow h_F}{F' \rightarrow F} \quad \begin{array}{c} \nearrow g \\ F'' \dashrightarrow F' \xrightarrow{f} h_F \\ \searrow f.g = f \circ g \end{array}$$