

Infinitesimal version of the Inverse Function Theorem

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The aim of this pamphlet is to prove the following

Theorem 0.1 *Let $f : D_\infty^n \rightarrow D_\infty^n$ be a function such that $f(0) = 0$ and $f^\bullet(0)$ is invertible. Then f is an isomorphism.*

Proof: We do first the case $n = 3$: Write $f = (f^1, f^2, f^3)$ and develop the f^i 's in Taylor series:

$$(*) \quad \begin{cases} f^1 = a_{(1,0,0)}^1 x_1 + a_{(0,1,0)}^1 x_2 + a_{(0,0,1)}^1 x_3 + a_{(2,0,0)}^1 x_1^2 + a_{(1,1,0)}^1 x_1 x_2 + \dots \\ f^2 = a_{(1,0,0)}^2 x_1 + a_{(0,1,0)}^2 x_2 + a_{(0,0,1)}^2 x_3 + a_{(2,0,0)}^2 x_1^2 + a_{(1,1,0)}^2 x_1 x_2 + \dots \\ f^3 = a_{(1,0,0)}^3 x_1 + a_{(0,1,0)}^3 x_2 + a_{(0,0,1)}^3 x_3 + a_{(2,0,0)}^3 x_1^2 + a_{(1,1,0)}^3 x_1 x_2 + \dots \end{cases}$$

We want to find $g = (g_1, g_2, g_3)$ such that $f \circ g = id$. By developing in Taylor series

$$(**) \quad \begin{cases} g_1 = b_{(1,0,0)}^1 x_1 + b_{(0,1,0)}^1 x_2 + b_{(0,0,1)}^1 x_3 + b_{(1,1,0)}^1 x_1 x_2 + \dots \\ g_2 = b_{(1,0,0)}^2 x_1 + b_{(0,1,0)}^2 x_2 + b_{(0,0,1)}^2 x_3 + b_{(1,1,0)}^2 x_1 x_2 + \dots \\ g_3 = b_{(1,0,0)}^3 x_1 + b_{(0,1,0)}^3 x_2 + b_{(0,0,1)}^3 x_3 + b_{(1,1,0)}^3 x_1 x_2 + \dots \end{cases}$$

Writing f and g in terms of its components, the composite $f \circ g$ may be written as

$$(***) \quad \begin{cases} f^1(g_1, g_2, g_3) = a_{(1,0,0)}^1 g_1 + a_{(0,1,0)}^1 g_2 + a_{(0,0,1)}^1 g_3 + \\ a_{(2,0,0)}^1 (g_1)^2 + a_{(1,1,0)}^1 g_1 g_2 + \dots \\ f^2(g_1, g_2, g_3) = a_{(1,0,0)}^2 g_1 + a_{(0,1,0)}^2 g_2 + a_{(0,0,1)}^2 g_3 + \\ a_{(2,0,0)}^2 (g_1)^2 + a_{(1,1,0)}^2 g_1 g_2 + \dots \\ f^3(g_1, g_2, g_3) = a_{(1,0,0)}^3 g_1 + a_{(0,1,0)}^3 g_2 + a_{(0,0,1)}^3 g_3 + \\ a_{(2,0,0)}^3 (g_1)^2 + a_{(1,1,0)}^3 g_1 g_2 + \dots \end{cases}$$

The equation $f \circ g = id$ may be re-written in terms of components as

$$f^i(g_1(x_1, x_2, x_3), g_2(x_1, x_2, x_3), g_3(x_1, x_2, x_3)) = x_i$$

The coefficient of $x_1 = x_1^{(1,0,0)}$ in the LHS is

$$a_{(1,0,0)}^1 b_{(1,0,0)}^1 + a_{(0,1,0)}^1 b_{(1,0,0)}^2 + a_{(0,0,1)}^1 b_{(1,0,0)}^3$$

The coefficient of $x_1^2 = x_1^{(2,0,0)}$ in the LHS is

$$a_{(1,0,0)}^1 b_{(2,0,0)}^1 + a_{(0,1,0)}^1 b_{(2,0,0)}^2 + a_{(0,0,1)}^1 b_{(2,0,0)}^3 + a_{(2,0,0)}^1 (b_{(1,0,0)}^1)^2 + a_{(1,1,0)}^1 b_{(1,0,0)}^1 b_{(1,0,0)}^2 + a_{(0,1,1)}^1 b_{(1,0,0)}^2 b_{(1,0,0)}^3 + a_{(1,0,1)}^1 b_{(1,0,0)}^3 b_{(1,0,0)}^1$$

The coefficient of $x_1 x_2 = x^{(1,0,0)} x^{(0,1,0)} = x^{(2,0,0)}$ in the LHS is

$$\left\{ \begin{array}{l} a_{(1,0,0)}^1 b_{(1,1,0)}^1 + a_{(0,1,0)}^1 b_{(1,1,0)}^2 + a_{(0,0,1)}^1 b_{(1,1,0)}^3 + \\ a_{(1,0,0)}^1 [b_{(1,0,0)}^1 b_{(0,1,0)}^2 + b_{(0,1,0)}^1 b_{(1,0,0)}^2] + \\ a_{(1,0,1)}^1 [b_{(1,0,0)}^1 b_{(0,1,0)}^3 + b_{(0,1,0)}^1 b_{(1,0,0)}^3] + \\ a_{(0,1,1)}^1 [b_{(1,0,0)}^2 b_{(0,1,0)}^3 + b_{(0,1,0)}^2 b_{(1,0,0)}^3] + \\ 2[a_{(2,0,0)}^1 b_{(1,0,0)}^1 b_{(0,1,0)}^1 + a_{(2,0)}^1 b_{(1,0,0)}^2 b_{(0,1,0)}^2 + a_{(0,0,2)}^1 b_{(1,0,0)}^3 b_{(0,1,0)}^3] \end{array} \right.$$

For the general case, the crucial observation is the following rather obvious

Lemma 0.2 *The coefficient of x^α in $f^j(g_1(x), \dots, g_n(x))$ is an expression of the form*

$$\sum_{i=1}^n a_{e_i}^j b_\alpha^i + \text{terms in } b_\beta^i$$

with $|\beta| < |\alpha|$. (Here, $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, with 1 in the i^{th} place.)

We now compare coefficients of $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ in both sides, with α the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. For instance, for $x^{(1,0,0)} = x_1^1 x_2^0 x_3^0 = x^1$,

$$\left\{ \begin{array}{l} 1 = a_{(1,0,0)}^1 b_{(1,0,0)}^1 + a_{(0,1,0)}^1 b_{(1,0,0)}^2 + a_{(0,0,1)}^1 b_{(1,0,0)}^3 \\ 0 = a_{(1,0,0)}^2 b_{(1,0,0)}^1 + a_{(0,1,0)}^2 b_{(1,0,0)}^2 + a_{(0,0,1)}^2 b_{(1,0,0)}^3 \\ 0 = a_{(1,0,0)}^3 b_{(1,0,0)}^1 + a_{(0,1,0)}^3 b_{(1,0,0)}^2 + a_{(0,0,1)}^3 b_{(1,0,0)}^3 \end{array} \right.$$

Since the Jacobian is invertible, $b_{(1,0,0)}^1, b_{(1,0,0)}^2, b_{(1,0,0)}^3$ are uniquely determined by these equations. Proceeding as before with the coefficients for x_2 (respectively x_3) we show, similarly, that $b_{(0,1,0)}^1, b_{(0,1,0)}^2, b_{(0,1,0)}^3$ (respectively

$b_{(0,0,1)}^1, b_{(0,0,1)}^2, b_{(0,0,1)}^3$) are also uniquely determined. We keep on going by identifying further coefficients. We do this by course of value induction: assume that all b_β^i are determined for all β with $|\beta| < |\alpha|$. Then b_α^i is determined. As an example: by comparing the coefficients of x_1^2 in (***) , we obtain

$$\begin{cases} a_{(1,0,0)}^1 b_{(2,0,0)}^1 + a_{(0,1,0)}^1 b_{(2,0,0)}^2 + a_{(0,0,1)}^1 b_{(2,0,0)}^3 = C_1 \\ a_{(1,0,0)}^2 b_{(2,0,0)}^1 + a_{(0,1,0)}^2 b_{(2,0,0)}^2 + a_{(0,0,1)}^2 b_{(2,0,0)}^3 = C_2 \\ a_{(1,0,0)}^3 b_{(2,0,0)}^1 + a_{(0,1,0)}^3 b_{(2,0,0)}^2 + a_{(0,0,1)}^3 b_{(2,0,0)}^3 = C_3 \end{cases}$$

with

$$-C_1 = a_{(2,0,0)}^1 (b_{(1,0,0)}^1)^2 + a_{(1,1,0)}^1 b_{(1,0,0)}^1 b_{(1,0,0)}^2 + a_{(0,1,1)}^1 b_{(1,0,0)}^2 b_{(1,0,0)}^3 + a_{(1,0,1)}^1 b_{(1,0,0)}^3 b_{(1,0,0)}^1$$

already determined. Similarly for C_2 and C_3 .

Since the Jacobian is invertible, $b_{(2,0,0)}^1, b_{(2,0,0)}^2, b_{(2,0,0)}^3$ are uniquely determined.

In a similar vein, by comparing coefficients of $x_1 x_2$ we can deduce that $b_{(1,1,0)}^1, b_{(1,1,0)}^2, b_{(1,1,0)}^3$ are uniquely determined too, etc.

In the general case, the previous lemma implies the conclusion: in fact, assume by course of value induction that all b_β^i 's are uniquely determined. By comparing coefficients in $f \circ g = id$, we have the system of equations to determine the coefficient of x_α , in $f^j(g_1, \dots, g_n)$:

$$\delta_\alpha^{e_j} = \sum_{i=1}^n a_{e_i}^j b_\alpha^i + \text{terms in } b_\beta^i$$

or, isolating the sum,

$$\sum_{i=1}^n a_{e_i}^j b_\alpha^i = C^j = \text{a constant}$$

since all the terms b_β^i 's are uniquely determined. But the Jacobian is invertible and this implies that all the b_α^j (all j) are uniquely determined. COMPLETE AND CHECK!