# Second-order DEs and geodesics

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## 1 Second order differential equations

*A second-order differential equation (DE) on a manifold* $M$ is a map $X : M^D \longrightarrow M^{D\times D}$ which is a section of the two maps $\rho_1, \rho_2 : M^{D\times D} \longrightarrow M^D$ induced by the two canonical inclusions of $D$ into $D \times D$. Equivalently, a second order DE on a manifold $M$ is a first-order DE on $M^D$ which is symmetric in the sense that $s \circ X = X$ (where $s$ is the map induced by the obvious twisting $tw : D \times D \longrightarrow D \times D$, i.e., $s = M^{tw}$). QUESTION: Does this equivalence still holds for microlinear spaces?

*A solution of a second order differential equation* $X$ is a map $\gamma : I \longrightarrow M$ such that $\gamma^\bullet : I \longrightarrow M^D$ is solution of $X$ considered as a first-order DE on $M^D$, i.e., $X \circ \gamma^\bullet = \gamma^{\bullet\bullet}$. An *initial condition of a solution* $\gamma$ is given by an equation $\gamma^\bullet(0) = V_0 \in M^D$, i.e., by an initial condition of the solution $\gamma^\bullet$ of $X$ (considered as a first-order DE). Thus, an initial condition for a solution $\gamma$ of a second order DE is given by the couple of conditions $\gamma(0) = x$ and $\gamma^\bullet(0) = V$ with $V \in M_x$.

This point of view on second-order DE results from the following considerations: consider the (autonomous) equation

$$y'' = a(y, y')$$

We may reduce the previous equation to a system of first-order DE by letting $z = y'$. In fact, the previous equation is equivalent to the system

$$\begin{cases} z = y' \\ z' = a(y, z) \end{cases}$$

This system may be identified with the vector field $Q : R^2 \longrightarrow (R^2)^D$ defined by

$$Q_d(y,z) = (y,z) + d(z, a(y,z))$$

By the identification $R^D = R \times R$ we have

$$Q(y,z) = ((y,z),(z,a(y,z))) = ((y,z), ze_1 + a(y,z)e_2)$$

or

$$Q_d(y,z) = z\partial_1|_{(y,z)}(d) + a(y,z)\partial_2|_{(y,z)}(d)$$

recalling that $\partial_i|_{(y,z)} = ((y,z),e_i)$, or, as it is written often in texts,

$$Q = \dot{q}_1 \partial/\partial q_1 + a(q_1,\dot{q}_1)\partial/\partial \dot{q}_1$$

In the general case (for arbitrary n) the system of first-order DE's corresponding to the second-order DE

$$q_i'' = a_i(q_j, q_j')$$

is

$$\begin{cases} q_i' = \dot{q}_i \\ \dot{q}_i{}' = a_i(q_j,\dot{q}_j) \end{cases}$$

and as a vector field it has the expression

$$Q = \sum_i (\dot{q}_i \partial/\partial q_i + a_i(q_j,\dot{q}_j)\partial/\partial \dot{q}_i)$$

Question: where does symmetry intervene?

Answer: by spelling the expression for the vector field $Q(y,z)$ we have

$$Q(y,z)(d_1,d_2) = y + d_1 z + d_2 z + d_1 d_2 a(y,z)$$

and this function is clearly symmetric (in $d_1$ and $d_2$).

Alternatively, we can proceed as follows: as just pointed out,

$$Q(y,z)(d_1,d_2) = y + (d_1+d_2)z + d_1 d_2 a(y,z)$$

On the other hand, if $\gamma$ is any curve on $M$,

$$\begin{aligned} \gamma^{\bullet\bullet}(t)(d_1,d_2) &= \gamma(t+d_1+d_2) \\ &= \gamma(t) + (d_1+d_2)\gamma'(t) + d_1 d_2 \gamma''(t) \end{aligned}$$

If $\gamma$ is a solution of $Q$, $\gamma^{\bullet\bullet}(t)(d_1, d_2) = Q(\gamma^{\bullet}(d_1, d_2)$. Comparing both expressions, $\gamma''(t) = a(\gamma(t), \gamma'(t))$.

In other words, $\gamma$ is solution of the 2d order DE:

$$y'' = a(y, y')$$

**Proposition 1.1** *Let $\Gamma : I \longrightarrow M^D$ be an integral curve of the DE (of second-order) $Q : M^D \longrightarrow (M^D)^D$. If $\gamma = \pi_M \circ \Gamma$, then $\gamma^{\bullet} = \Gamma$.*

*Proof:* From $\gamma(t) = \Gamma(t)(0)$, we obtain

$$\begin{aligned} \gamma^{\bullet}(t)(d) &= \gamma(t + d) \\ &= \Gamma(t + d)(0) \\ &= \Gamma^{\bullet}(t)(d)(0) \\ &= Q(\Gamma(t))(d)(0) \\ &= Q(\Gamma(t))(0)(d) \\ &= \Gamma^{\bullet}(t)(0)(d) \\ &= \Gamma(t)(d) \end{aligned}$$

The fifth line is a consequence of symmetry (of $Q$).

NB Although not needed, notice that

(1) $Q(\Gamma(t)) \in M^D_{\Gamma(t)}$
(2) For every $t \in I,$ there are uniquely defined $q^1(t),\ q^2(t) \in M_{\Gamma(t)}$ such that $Q(\Gamma(t))(d_1) = q^1(t) + d_1 q^2(t)$

As to the first: $Q(\Gamma(t))(d_1)(d_2) = \Gamma^{\bullet}(t)(d_1)(d_2)$, $Q(\Gamma(t))(d_2)(d_1) = \Gamma^{\bullet}(t)(d_1)(d_2)$, by symmetry. For $d_1 = 0$, $Q(\Gamma(t))(d_2, 0) = \Gamma^{\bullet}(t)(0)(d_2) = \Gamma(t)(d_2)$ for every $d_2 \in D$.

The second is a consequence of microlinearity of $M$ (and consequently the K/L axiom holds for fibers).

**Corollary 1.2** *Let $I \subset R$ be open and $Q$ a second order DE. Then the couple $((-)^{\bullet},\ \pi_M \circ (-))$ establishes a 1-1 correspondence between solutions of the second order differential equation $Q$ on $M$ defined on $I$ and integral curves of $Q$ (considered as a vector field on $M^D$) with domain $I$.*

## 2 Geodesics

Recall that a geodesic on a manifold $M$ with a symmetric connection $M$ is a curve $\gamma : R \longrightarrow M$ such that $\gamma^{\bullet\bullet} = \nabla(\gamma^\bullet, \gamma^\bullet)$.

As the notation suggests, $\gamma$ is a solution of a second-order DE. In fact, define $G : M^D \longrightarrow M^{D\times D}$ by $G(v) = \nabla(v, v)$. We claim the $G$ is a second order differential equation. To see this, we first check that $G$ is a vector field on $M^D$. This is obvious, since $G(v)(d, 0) = \nabla(v, v)(d, 0) = v(d)$. The other condition, namely that $M^{tw} \circ G = G$ follows from the symmetry of the connection: $G(v)(d_1, d_2) = \nabla(v, v)(d_1, d_2) = \nabla(v, v)(d_2, d_1) = G(v)(d_2, d_1)$.

In the sequel, we restrict ourselves to infinitesimal curves, i.e., curves with domain $D_\infty$.

**Proposition 2.1** *Let $u \in M_x$ and $X : M^D \longrightarrow M^{D\times D}$ be a second order differential equation on $M$. Then $X$ has a unique solution $\gamma : D_\infty \longrightarrow M$ with $\gamma(0) = x$ and $\gamma^\bullet(0) = u$.*

*Proof:* Particular case of the existence of unique formal solutions of the first-order DE $X$ on the manifold $M^D$.

To see what's going on, consider the particular case $M = R$ and the DE

$$Y : \; y'' = f(y, y')$$

This is equivalent to the couple of first order DE

$$Y : \begin{cases} y' = z \\ z' = f(y, z) \end{cases}$$

In the usual notation

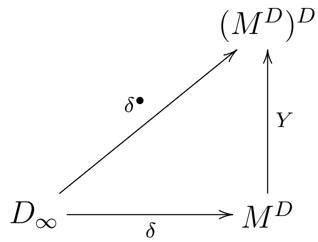
$$\begin{cases} (y^1)' = f^1(y^1, y^2) \\ (y^2)' = f^2(y^1, y^2) \end{cases}$$

with $f^1(y, z) = z$, $f^2(y, z) = f(y, z)$.

By applying the theorem on the existence of unique formal solutions of first order DE's, we have

$$\delta : D_\infty \longrightarrow M^D$$

such that the diagram




is commutative and $\delta(0) = u$.

Let $\gamma = \delta^1$. We claim that $\delta = \gamma^\bullet$.

In fact,

$$\begin{aligned}
\gamma^\bullet(t)(d) &= (\delta^1)^\bullet(t)(d) \\
&= \delta^1(t+d) \\
&= \delta^1(t) + d(\delta^1)'(t) \\
&= \delta^1(t) + d\delta^2(t) \\
&= \delta(t)(d)
\end{aligned}$$

(since $\delta$ is a solution of $Y$, $(\delta^1)'(t) = \delta^2(t)$, hence the 4th line)

This shows that $\gamma : D_\infty \longrightarrow M$ is a solution of $Y$. Let $\alpha$ and $\beta$ be solutions of $Y$ with same initial conditions $(x, u)$. Then $\alpha^\bullet = \beta^\bullet$ by uniqueness of solutions of first order DE on $M^D$. This implies that $\alpha = \beta$.

# 3 Existence of enough geodesic fields

The aim of this section is to prove the theorem of the title. In more details,

**Proposition 3.1** *Let $M$ be a $n$-dimensional $D_\infty$-manifold and $u \in M_x$. Then there is an infinitesimal neighborhood of $x$ (i.e., isomorphic to $D_\infty^n$) with $u \in V^D$ and a geodesic field $Q : V \longrightarrow V^D$ such that $Q(x) = u$.*

NB A *$n$-dimensional $D_\infty$-manifold* $M$ is a space in which every point $p$ has an infinitesimal neighborhood $U \ni p$, i.e., a subset $U \subset M$ with $p \in U$ such that $U$ is isomorphic to $D_\infty^n$. (Here $D_\infty$=nilpotents of $R$).

*Proof:* For the sake of simplicity of notation assume $n = 2$.

Cover $M$ by infinitesimal neighborhoods, i.e., $M = \bigcup_\alpha V_\alpha$. Since $D$ is an atom, $M^D = \bigcup_\alpha V_\alpha^D$. Thus, $u \in V_\alpha$ for some $\alpha$. Let $V = V_\alpha$.

Without loss of generality, we may assume that $V = D^2_\infty$. We may further assume (by choosing appropriate coordinates) that $x = (0,0)$ and $u = (0,0,1,0)$, i.e., $u$ is the horizontal unitary vector starting at $(0,y)$. Notice that $(D^2_\infty)^D = D^2_\infty \times R^2$ (by KL).

Define a function $\phi : D^2_\infty \longrightarrow D^2_\infty$ as follows: $\phi(t,y) = \gamma_y(t)$, where $\gamma_y : D_\infty \longrightarrow D^2_\infty$ is the geodesic with initial conditions $\gamma_y(0) = (0,y)$ and $\gamma^\bullet(0) = (0,y,1,0)$. (NB Existence and uniqueness follow from previous section on geodesics).

**Claim: $\phi$ is a bijection.**

*Proof:* Write $\phi = (\phi_1, \phi_2)$. By re-writing the initial condition $\gamma_y(0) = (0,y)$, in terms of the $\phi$'s, $\phi_1(0,y) = 0$ and $\phi_2(0,y) = y$. Similarly, the second initial condition $\gamma^\bullet_y(0) = (0,y,1,0)$ may be re-written as $(\partial\phi_1/\partial t)(0,y) = 1$ and $(\partial\phi_2/\partial t)(0,y) = 0$.

On the other hand, from $\phi_1(0,y+d) = \phi_1(0,y) + d(\partial\phi_1/\partial y)(0,y)$ together with the first initial condition we get $(\partial\phi_1/\partial y)(0,y) = 0$. Similarly using the corresponding equation for $\phi_2$, $(\partial\phi_2/\partial y)(0,y) = 1$.

Hence the Jacobian of the transformation $\phi$ at $(0,y)$ is the unity and $\phi$ has an inverse, by the inverse function theorem for functions on $D^n_\infty$. ("Infinitesimal Inverse Function Theorem.pdf").

We finally define $Q : D^2_\infty \longrightarrow D^2_\infty \times R^2$ by the prescription $Q(t,y) = \gamma^\bullet_x(s)$ where $\phi(s,x) = (t,y)$.

**Claim 1:** $Q$ is a vector field and $Q(0,0) = (0,0,1,0)$. In fact, $Q(t,y)(0) = \gamma^\bullet_x(s)(0) = \gamma_x(s) = \phi(s,x) = (t,y)$. Furthermore, $Q(0,0) = \gamma^\bullet_0(0)$. Thus, the second initial condition gives in fact $Q(0,0) = (0,0,1,0)$.

**Claim 2:** The $\gamma_s$'s are integral curves of $Q$, i.e., $\gamma^\bullet_s(t) = Q(\gamma_s(t)$. This is clear: $\gamma_s(t) = \phi(t,s) = (\phi_1(t,s), \phi_2(t,s))$. Therefore, by definition of $Q$, $Q(\gamma_s(t)) = \gamma^\bullet_x(t')$ with $\phi(t',x) = (\phi_1(t,s), \phi_2(t,s))$. Since $\phi$ is bijective, $t = t', s = x$. This implies the conclusion.

**Claim 3:** The integral curves of $Q$ are precisely those of the form $\gamma(t) = \gamma_{x_1}(t_1 + t)$, for $(t_1, x_1) \in D_\infty \times N$. In fact, all curves of this form are integral curves of $Q$ as the following argument shows: let $u(t) = t_1 + t$. Then $\gamma^\bullet \circ u =$

$(\gamma \circ u)^\bullet$. Indeed, computing the first term at $(t, d)$ with $d \in D$, we have

$$\begin{aligned}
\gamma^\bullet(u(t))(d) &= \gamma(u(t) + d) \\
&= \gamma((t_1 + t) + d) \\
&= \gamma(t_1 + (t + d)) \\
&= \gamma(u(t + d)) \\
&= (\gamma \circ u)(t + d) \\
&= (\gamma \circ u)^\bullet(t)(d)
\end{aligned}$$

Since $\gamma^\bullet = Q \circ \gamma$, then $(\gamma \circ u)^\bullet = \gamma^\bullet \circ u = Q \circ (\gamma \circ u)$.

To prove the converse, let $\gamma$ be an integral curve of $Q$ with initial condition $\gamma(0) = (t_0, x_0)$. Since $\phi$ is bijective, there is a unique couple $(t_1, x_1)$ such that $\phi(t_1, x_1) = (t_0, x_0)$. Define $\delta(t) = \gamma_{x_1}(t_1 + t)$. Then, $\delta = \gamma$ since both are integral curves of $Q$ with the same initial condition: $\delta(0) = \gamma_{x_1}(t_1) = \phi(t_1, x_1) = (t_0, x_0) = \gamma(0)$.

**Corollary 3.2** *The vector field $Q$ is geodesic, i.e., all integral curves of $Q$ are geodesics.*

*Proof:* We have to show that if $\gamma$ is a geodesic, so is $\delta$, where $\delta(t) = \gamma(t_0 + t)$.

Define $u(t) = t_0 + t$. Thus, $\delta = \gamma \circ u$. Using what we proved above, i.e., $\gamma^\bullet \circ u = (\gamma \circ u)^\bullet$. Therefore $\gamma^{\bullet\bullet} \circ u = (\gamma^\bullet \circ u)^\bullet = (\gamma \circ u)^{\bullet\bullet}$.

But from $\gamma^{\bullet\bullet} = \nabla(\gamma^\bullet, \gamma^\bullet)$ we obtain, by post-composing with $u$, $\gamma^{\bullet\bullet} \circ u = \nabla(\gamma^\bullet, \gamma^\bullet) \circ u$, i.e., $(\gamma \circ u)^{\bullet\bullet} = \nabla(\gamma^\bullet, \gamma^\bullet) \circ u$.

On the other hand,

$$\begin{aligned}
\nabla(\gamma^\bullet, \gamma^\bullet) \circ u &= (\nabla \circ (\gamma^\bullet, \gamma^\bullet)) \circ u \\
&= \nabla \circ ((\gamma^\bullet, \gamma^\bullet) \circ u) \\
&= \nabla \circ (\gamma^\bullet \circ u, \gamma^\bullet \circ u) \\
&= \nabla(\gamma^\bullet \circ u, \gamma^\bullet \circ u)
\end{aligned}$$

Thus, $\delta^{\bullet\bullet} = \nabla(\delta^\bullet, \delta^\bullet)$ and this concludes the proof.