Straightening out Theorem

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1 Straightening out Theorem

The aim of the pamphlet is to prove the the following

Theorem 1.1 (Straightening out theorem) *Let Q be a vector field on a n-dimensional D_∞-manifold M and $p \in M$ such that $Q(p) \neq 0$. Then there are coordinates ϕ around p making the following diagram commutative*

$$\begin{array}{ccc} (D_\infty^n)^D & \xrightarrow{\phi^D} & M^D \\ \uparrow{\scriptstyle \partial_1} & & \uparrow{\scriptstyle Q} \\ D_\infty^n & \xrightarrow[\phi]{} & M \end{array}$$

with $\phi(0) = p$.

NB We recall that a D_∞-*manifold of dimension* n is a space in which every point has an infinitesimal neighborhood, i.e. a neighborhood isomorphic to D_∞^n.

Proof: Since the question is infinitesimal (rather than local), we may assume that $M = D_\infty^n$. Furthermore, we may assume that $p = 0$ and $Q(0) = \partial_1|_0$. Through each point of $(0, \delta_2, \ldots, \delta_n)$ passes a unique integral curve of Q. Define $\xi : D_\infty \longrightarrow M$ by $\xi(\delta_1, \ldots, \delta_n) = Q_{\delta_1}(0, \delta_2, \ldots, \delta_n)$. Thus, Q_{δ_1} is the integral curve of Q starting at $(0, \delta_2, \ldots, \delta_n)$.

Claim:

$$\begin{cases} \xi^D \circ \partial_1 = Q \circ \xi \\ (\xi^D \circ \partial_i)(0) = \partial_i|_0 \ \text{ for } i > 1 \end{cases}$$

Proof: Simple computations

$$\begin{array}{rcl} (\xi^D \circ \partial_1)(\delta)(h) & = & \xi^D(\partial_1|_\delta)(h) \\ & = & \xi(\partial_1|_\delta(h)) \\ & = & \xi(\delta^1 + h, \delta^2, \ldots, \delta^n) \\ & = & Q_{\delta^1+h}(0, \delta^2, \ldots, \delta^n) \\ & = & Q_h(Q_{\delta^1}(0, \delta^2, \ldots, \delta^n)) \\ & = & Q_h(\xi(\delta)) \\ & = & Q(\xi(\delta))(h) \\ & = & (Q \circ \xi)(\delta)(h) \end{array}$$

As for the other (for $i > 1$),

$$\begin{array}{rcl} (\xi^D \circ \partial_i)(0)(h) & = & \xi^D(\partial_i|_0)(h) \\ & = & \xi(\partial_i|_0(h)) \\ & = & \xi(0, \ldots, h, 0, \ldots, 0) \\ & = & Q_0(0, \ldots, h, 0, \ldots, 0) \\ & = & (0, \ldots, h, 0, \ldots, 0) \\ & = & \partial_i|_0(h) \end{array}$$

Since $\xi^D(\partial_1|_0) = (\xi^D \circ \partial_1)(0) = (Q \circ \xi)(0) = Q(\xi(0)) = Q(0) = \partial_1|_0$, the derivative at 0, i.e., the linear transformation

$$\xi'(0) = \xi^D_0 : M^D_0 \longrightarrow M^D_0$$

is the identity. By the Inverse Function Theorem, ξ has an inverse ϕ which defines coordinates x around 0. The first claim implies that $Q_x = \partial_1$, as the following formal equivalences show

$$\frac{\dfrac{\xi^D \circ \partial_1 = Q \circ \xi}{\partial_1 = (\xi^{-1})^D \circ Q \circ \xi}}{\partial_1 \circ \xi^{-1} = (\xi^{-1})^D \circ Q}$$

Geometrically, the idea is very simple. To simplify, take $n = 2$. Since the integral curves of ∂_1 are horizontal lines in the plane, we straighten out the integral lines of Q so that they become horizontal. The transformation $\phi = \xi^{-1}$ does precisely this.