We define our new reduced theory to be $T'' = T' \cap \Sigma'$.

By passage to quotients, $\mathscr{J}'$ defines a family $\langle I'_n : n \in S_\omega(V)\rangle$ of boolean isomorphisms $I'_n : B_\models(\Sigma', T'')(n) \simeq B_\models(\Sigma, T)(n)$, whose inverse satisfies $I_n[\phi] = [\mathscr{J}(\phi)]$ for all $\phi \in \Sigma^n$.

If $\mathfrak{A}'$ is a model of T'' with universe A, $\mathfrak{A}'$ defines families of boolean homomorphisms $\alpha'_n : B_\models(\Sigma', T'')(n) \to 2^{A^{|V|}}$ and $\alpha_n : B_\models(\Sigma, T)(n) \to 2^{A^{|V|}}$, as before. Since α'_n is a Σ'-homomorphism, α_n preserves all the (countable) sums of the type $\{[VE[\phi'_\xi]] : \xi \in \mu'\}$ for $E \in \mathscr{E}\{\phi'_\xi : \xi \in \mu'\}$. By our version of Sikorski's theorem, α_n preserves all the disjoint sums of the form $\{[\phi'_\xi] : \xi \in \mu'\}$. A fortiori, α_n preserves all sums of the form $\{[\phi_\xi] : \xi \in \mu'\}$ such that $V\langle \phi_\xi : \xi \in \mu'\rangle$ is in Σ_0 and is a semantical consequence of T, i.e., α_n is a Σ_0-homomorphism. If, just as before, we let $\mathfrak{A}$ be the structure obtained from $\mathfrak{A}'$ via $\mathscr{J}$, we easily show (by induction) that $\alpha_n[\phi] = \phi^{\mathfrak{A}}$, for all $\phi \in \Sigma_0$. This implies that $\mathfrak{A}$ is a model of T. The rest of the proof follows as in the former case.

From the property of unique readability of our languages, it follows that $\mathscr{J}'$ is an injective map. This implies that $|\Sigma'| \le |\Sigma|$. An easy computation gives the following bound for Σ which is, a fortiori, a bound for T'':

$$|T''| \le |\Sigma| \le |T| \cdot |V_L| \cdot \sum_{\nu < \mu} m(\nu).$$

§3. In this last section we give examples of theories in $L_{\mu^+\omega}$ languages for which the cardinality bounds for the reduced theories are actually attained. As a corollary, we obtain the converse of our main theorem.

Let μ be an infinite cardinal and let L_{μ^+} be the $L_{\mu^+\omega}$ language generated by the binary relation symbol $\le$ and the set of unary relation symbols $\{P_\xi : \xi \in \mu\}$. We assume that $V_L = \{x_\xi : \xi \in \mu\}$. By induction, we define $\phi_\xi(x_1) \in L_{\mu^+}$ such that $(\mu, \le) \models \phi_\xi[\eta]$ iff $\xi = \eta$. (Actually, we can obtain such a ϕ_ξ in the $L_{\mu\omega}$ language L_μ generated by these primitive symbols.)

Let σ_μ be the theory in the language L_{μ^+} consisting of the following sentences:

"$\le$ is a linear ordering",
$\Lambda\langle \forall x_1(P_\xi(x_1) \leftrightarrow \phi_\xi(x_1)) : \xi \in \mu\rangle$,
$\Lambda\langle \exists x_1 P_\xi(x_1) : \xi \in \mu\rangle$,
$\forall x_1 V\langle P_\xi(x_1) : \xi \in \mu\rangle$.

The theory σ_μ is categorical and characterizes $\langle \mu, \le, \{\xi\}\rangle_{\xi\in\mu}$ up to (unique) isomorphism.

THEOREM. *If σ_μ is equivalent to some theory T in some $L_{\omega_1\omega}$ language L, then $\mu < \theta$ and $|T| \ge m(\mu)$.*

PROOF. We let $\mathscr{J}$ (resp. $\mathscr{J}'$) be the map from the atomic formulas of L_{μ^+} (resp. L) into the formulas of L (resp. L_{μ^+}). By hypothesis, these maps establish a one-to-one correspondence between models having μ as universe. In particular, if $\mathfrak{A}$ is the model obtained from $\langle \mu, \le, \{\xi\}\rangle_{\xi\in\mu}$ via $\mathscr{J}'$, T characterizes $\mathfrak{A}$ up to unique isomorphism.

Since $\langle \mu, \le, \{\xi\}\rangle_{\xi\in\mu}$ is obtained itself from $\mathfrak{A}$ via $\mathscr{J}$, every $\xi \in \mu$ is definable in $\mathfrak{A}$ by $P'_\xi(x_1) = \mathscr{J}(P_\xi(x_1))$. Furthermore, the well-ordering $\le$ on μ is definable in $\mathfrak{A}$ by $x_1 \le' x_2 = \mathscr{J}(x_1 \le x_2)$.