Let Σ be the fragment of L generated by T, $\{P'_\xi(x_1): \xi \in \mu\}$ and $x_1 \leq' x_2$. We notice that Σ contains all the formulas of type

$$(*) \qquad \exists x_1, \cdots, \exists x_k(\theta \wedge \phi), \quad \text{whenever } \theta, \phi \in \Sigma.$$

Since T is categorical, $|T| \geq \mu$ by the Lowenheim–Skolem theorem of [K, p. 115]. This implies that $|T| = |\Sigma|$.

Let E be the set of Σ-homomorphisms of $B_F(\Sigma, T)(1)$ into 2. Every disjoint (countable) sum of the form $\{[\phi_\xi \wedge \neg V\langle\phi_\eta: \eta \in \xi\rangle]: \xi \in \nu\}$ such that $V\langle\phi_\xi: \xi \in \nu\rangle \in \Sigma$ defines an equivalence relation on E and the set $\mathscr{B}$ of finite intersections of these equivalence relations forms a basis for a uniformity. A Cauchy filter-base determines in a canonical fashion a point of E to which it converges, i.e., E is complete. With the induced topology, E is the projective limit of the discrete spaces $\langle E/R: R \in \mathscr{B}\rangle$ and hence a closed subset of $\pi\langle E/R: R \in \mathscr{B}\rangle$ which is, in turn, a closed subset of N^Σ. We shall prove that E is a discrete space of cardinality μ.

Let $h \in E$. Then h defines an ultrafilter $\tilde{h}$ of the boolean algebra $D(2^\mu)$ of subsets of μ which are definable in Σ, i.e., $A \in D(2^\mu)$ iff for some $\phi(x_1) \in \Sigma$, $A = \{\xi \in \mu: \mathfrak{A} \models \phi[\xi]\}$. In particular, notice that every $\{\xi\} \in D(2^\mu)$. Furthermore, let $D(\mu^\mu)$ be the set of functions which are definable in Σ in the sense that $f \in D(\mu^\mu)$ if there is $\theta(x_1, x_2) \in \Sigma$ such that $f(\xi) = \eta$ iff $\mathfrak{A} \models \theta(x_1, x_2)[\xi, \eta]$. Since every $\{\xi\} \in D(2^\mu)$, the constant functions c_ξ belong to $D(\mu^\mu)$.

We define a Skolem ultrapower $\mathfrak{A}^*$ of Σ-definable functions in the usual manner: If $P \subset \mu^k$ is the interpretation of some primitive symbol of L in $\mathfrak{A}$, $(f_1, \cdots, f_k) \in P^*$ iff $\{\xi: \langle f_1(\xi), \cdots, f_k(\xi)\rangle \in P\} \in \tilde{h}$. The theorem of Łos can be proved (by induction on formulas of Σ) in the following form: $\mathfrak{A}^* = \phi(x_{\xi_1}, \cdots, x_{\xi_k})[f_{\xi_1}, \cdots, f_{\xi_k}]$ iff $\{\xi: \mathfrak{A} \models \phi[f_{\xi_1}(\xi), \cdots, f_{\xi_k}(\xi)]\} \in \tilde{h}$ whenever $\phi \in \Sigma, f_{\xi_1}, \cdots, f_{\xi_k} \in D(\mu^\mu)$.

In fact, we first notice that functions can be eliminated from the satisfaction relation in favor of their definitions, using the fact that the formulas of type (*) are in Σ. To handle the existential clause (in the inductive step), we notice that the usual well-ordering $\leq$ is definable in $\mathfrak{A}$ by $\mathscr{J}(x_1 \leq x_2)$.

As a corollary, $\mathfrak{A}^*$ is a model of T and this implies that $\mathfrak{A}^* \cong \mathfrak{A}$ for a unique isomorphism, taking id$/\tilde{h}$ into some $\xi \in \mu$. Therefore id$/\tilde{h}$ satisfies $P'_\xi(x_1)$ and this implies that $\{\xi\} \in \tilde{h}$. It is easily checked that the application $h \to \bigcap\tilde{h}$ is a bijection whose inverse is the map e defined by $e(\xi)([\phi]) = 1$ iff $\mathfrak{A} \models \phi[\xi]$, for all $\phi(x_1) \in \Sigma$. Since $\{e(\xi)\} = \{h: h[P'_\xi(x_1)] = 1\}$ is open, E is discrete.

We have shown that N^Σ embeds μ as a discrete closed subset. Therefore $\mu < \theta$ and $|T| = |\Sigma| \geq m(\mu)$.

COROLLARY. *Assume that every theory in a $L_{\mu\omega}$ language is reducible to some theory in a $L_{\omega_1\omega}$ language. Then $\mu \leq \theta$.*

When μ is not a successor cardinal, we do not know of examples of theories in $L_{\mu\omega}$ languages for which the cardinality bounds for the reduced theories are attained. It might be that the $L_{\mu\omega}$ theory of $\langle\mu, \leq, \{\xi\}\rangle_{\xi\in\mu}$ is already such an example.

Nonessential variations of this reduction theorem seem to be true for more general infinitary languages. In fact, the reduction of the infinitary propositional part was carried "locally", i.e., for each $n \in S_\omega(V)$ but in a "coherent" fashion (via the canonical isomorphism I_n). A more general statement and perhaps a more illuminating discussion of our theorem can be given in a categorical context, by