Models for Non-Boolean Negations in Natural Languages Based on Aspect Analysis

M. La Palme Reyes, J. Macnamara,
G. E. Reyes and H. Zolfaghari

May 1995

Abstract

Since antiquity two different negations in natural languages have been noted: predicate negation ('not honest') and predicate term negation ('dishonest'). Aristotle tried to formalize them in his system of oppositions, distinguishing between affirmation and negation ('honest' and 'not honest') and contraries ('honest' and 'dishonest'). The Stoics replaced Aristotle's logic of terms by their logic of propositions. Although they considered three types of negation, none of them corresponded to Aristotle's predicate term negation. Frege and modern logic have followed the Stoics in either identifying predicate term negation with predicate negation or in casting predicate term negation out of logic into the realm of pragmatics. Although an extensive literature has arisen on these issues, we have not found mathematical models. We propose category-theoretic models with two distinct negation operations, neither of them in general Boolean. We study combinations of the two ('not dishonest') and sentential counterparts of each. We touch briefly on quantifiers and modalities. The models are based on an analysis of aspects. For instance, to give an overall, global judgement of John's honesty we must agree on what aspects of John are relevant for that judgement: John qua person (global aspect), John qua social being (social aspect), John qua family man, John qua professional man, etc. We conceptualize this 'Aristotelian' analysis by means of a category of 'aspects'. A model (for the negations) is obtained from the category of presheaves on this category. Although neither of the negations

is Boolean, predicate negation turns out to be Boolean at the 'global' aspect (the aspect of the overall judgement) which may help to explain the persistent belief that logic is naturally Boolean.

Introduction

In natural languages such as English there are some common forms of negation that resemble little the single negation of classical logic. For example, we can deny that John is honest either by asserting that John is not honest or by asserting that John is dishonest. Morphologically, the first negates the predicable 'to be honest', the second the adjective 'honest'. Following the usual terminology we call the first 'predicate negation' and the second 'predicate term negation' (both at the syntactical and the semantical level). Intuitively, predicate term negation ('dishonest') is stronger than predicate negation ('not honest'). In fact, although John is either honest or not honest, it is not contradictory to assert that he is neither honest nor dishonest. Even languages like Chinese, which lack morphological means to express some of these negations, still have contrasts (as in English) between such predicates as 'good' and 'bad', 'healthy' and 'sickly', the second adjective of each pair being stronger than the predicate negation of the first. Notice that even English does not have a uniform lexical way to obtain strong negations of adjectives. Sometimes 'un' is used (as in 'happy', 'unhappy'), but 'unmoral' is not the strong negation of 'moral'. It is applied for instance to children, to indicate that they are outside the realm of morality. The reader may consult [9] for further discussion.

The lack of a classical counterpart for predicate term negation is readily apparent when one tries to translate a portion of natural language into a system of classical logic, whether first-order, higher-order, one-sorted or many-sorted. In usual formalizations, if $h(x)$ translates the predicate 'to be honest' applied to x, then $\neg h(x)$ is taken to be the translation of 'to be not honest' as applied to x. But how should we translate 'to be dishonest'? There is simply no logical operation of the formal system corresponding to the non-Boolean predicate term negation.

These observations, some of which are known at least since Aristotle's time (see section 1), raise the question of formalizing portions of natural languages in a system capable of accommodating both kinds of negation, a system whose semantics will describe the relations between them in a way that agrees with our intuitions about their relative strength and with the fact that they are morphologically related: the relation between 'being honest' and 'being dishonest' is certainly quite different from that between 'being a cat' and 'being a dog'.

For a number of years, the authors of this paper have been developing 'a theory of kinds' whose main novelty is the use of the mathematical theory of categories to capture what they believe is the most fundamental feature of human language: reference to reality, including linguistic reality (see [15], [8], [11]). Reference appears as a functor between the category of nouns and the category of kinds. This paper should be considered as an application of the theory of kinds. To simplify the exposition of the main ideas, however, we have replaced kinds by sets. Although it is not the first that they have devoted to non-Boolean negations (see below), the main novelty of this paper with respect to previous studies (including their own) is the conceptualization of the notion of 'qua' or 'aspect' which occurs in expressions like 'John is honest qua father, but not so qua politician', which are perfectly understandable to all but some philosophers! This formalization is obtained as a particular case of a well-known categorical construction, the so called 'comma category' of count nouns under a given count noun (see e.g. [14]). The models for negations turn out to be interpretations of some categories of aspects, i.e., functors from categories of aspects into the category of sets, as explained in detail in the text.

The body of this paper comprises three sections and one appendix. The first section is an historical survey of the thinking about negation from Aristotle on and relies on the monumental work of Horn [5] and on our previous publications [9], [10]. The second section concerns the conceptualization of aspects as a category. The third section introduces interpretations as (contravariant) functors from this category of aspects into the categories of sets and shows how to compute truth values under a given aspect. Modalities and quantifiers are also discussed (another novelty with respect to the authors' previous publications). A subsection deals with the unification of term negations with negations of propositions. The appendix, mainly mathematical, is devoted to people with some background in categorical logic, or at least in Kripke semantics, and places the work in the more general context of a presheaf topos on a pre-ordered set.

1 Historical Notes

When Aristotle invented logic what he invented was a logic of terms; and his categories were categories of terms. The Stoics, however, replaced Aristotle's term variables with propositional ones, and with that propositional logic was born. (See Lukasiewicz [13, page 199]).

For a long time term logic and propositional logic existed together. For example, William of Ockham [17] and [18] devoted the first part of his *Summa logicae* to terms and the second part to propositions. Perhaps it was Kant who is responsible for the triumph of propositional logic and the eclipse of term logic. For where Aristotle had categories of objects and attributes, closely related to the grammatical categories of terms that denote them, Kant has categories of concepts. These Kant derives from categories of judgements; that is from categories of propositions. With the move to categories of judgements term logic in anything like Aristotle's sense drops from view. In this what is now called 'classical logic' from Frege on follows Kant.

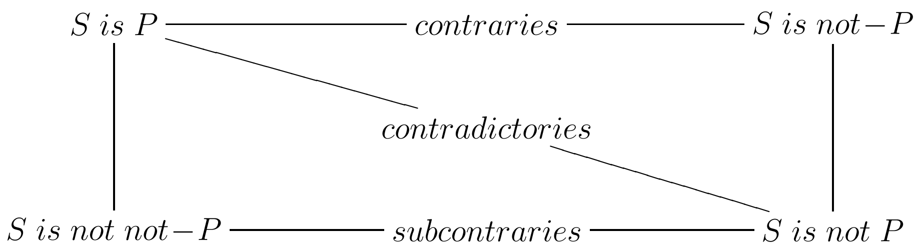
This paper will concentrate on two notions of Aristotle's term logic that did not survive the passage from term logic to the logic of propositions (at least in the sense that Aristotle understood them): predicate term negation and the preposition *qua*.

In Aristotle's term logic there are two negations, both applying to terms and neither applying to propositions. One gives rise to contrariety. For example, 'John is happy' and 'John is unhappy' are contraries (although Aristotle used an expression that is best translated as 'not-happy'). This means that both cannot be true; although both may be false: one is not necessarily either happy (a positive state) or unhappy (a quite negative state). The operator that transforms 'happy' into 'unhappy' is called the predicate term negation.

The second negation gives rise to contradictories. For example, 'John is happy' and 'John is not happy' are contradictories. This means that one is true precisely when the other is false. The operator that transforms 'happy' into 'not happy' is called the predicate negation.

Certain combinations of these negations give rise to subcontraries. For example 'John is not unhappy' and 'John is not happy' are subcontraries. This means that both cannot be false, although both may be true.

It is customary to represent these oppositions by using a square that Horn [5, page 16] renders as follows:



In his study of contrariety, Aristotle saw that the opposition between, for instance, 'odd' and 'even' (for numbers) is rather different from that between 'good' and 'bad'. On the one hand, he remarks that there is in numbers no mediate (intermediate) between 'odd' and 'even', a number is either odd or even. 'Odd' and 'even' are immediate contraries. On the other hand, he says that:

> ...bad and good are predicated both of men and of many other things, but it is not necessary for one or the other of them to belong to those things they are predicated of (for not all are either bad or good). And between these there is certainly something intermediate... In some cases there exist names for the intermediates, as with grey and yellow between white and black; in some, however, it is not easy to find a name for the intermediate, but it is by the negation of each of the extremes that the intermediate is marked off, as with the neither good nor bad and neither just nor unjust. (*Cat.* 12a 15-25)

As we mentioned, there is just no negation as an operation on propositions in Aristotle's term logic. On the other hand, it is not surprising that the Stoics, who replaced term logic by propositional logic, were the first to introduce an external operator on propositions transforming 'John is honest' into 'not: John is honest'. Although they distinguished three types of negation, none of them corresponds exactly to Aristotle's predicate negation (see [9]).

In modern times, Frege has been very influential on the way logic considers natural languages and for that matter on what is regarded as the domain of logic in natural languages. It is therefore regrettable that he has casted out from logic some chapters of Aristotle's term logic. Yet many of the phenomena that he considers to be outside the realm of logic are amenable to logic treatment. For example, he identifies predicate negation (and predicate term negation) with external propositional negation, as though logic were incapable of dealing with them as distinct phenomena.

Despite Frege's refusal of a place in logic for the subject-predicate distinction (as made by the grammarians) and contrariety, some philosophers have recognized the necessity of handling these phenomena and have tried to incorporate them in a logical system. For instance, Sommers [23], [24] and Englebretsen [3] worked on a term logic; McCall [16] and Rescher [19] have studied contrariety in the context of propositional and term logics (see [5, page 43]. None of these authors, however, has proposed mathematical models. In our paper [9], we have discussed McCall's approach. The readers interested may refer to that paper for more historical comments.

Turning to the study of the preposition *qua* (or *as*), let us observe that, as used by Aristotle, it becomes a term only when combined with certain terms A and B to form the new term A *qua* B (or A *as* B). This preposition plays a fundamental rôle in Aristotle's philosophy. In Book II of his *Physics*, for instance, he asserts:

> While geometry investigates natural line but not *qua* natural, optics investigates mathematical lines, but *qua* natural, not *qua* mathematical. (*Phy.* 194a 10-11)

In Book IV of his *Metaphysics*, Aristotle even characterizes first philosophy as the science that studies being *as* being:

> There is a science which investigates being as being and the attributes which belong to this in virtue of its own nature. (*Met.* 1003a 22-23)

An important point to be kept in mind to understand this expression was made by J. Barnes [2]:

> This is not an adjectival phrase modifying the word 'being': it is an adverbial phrase modifying the word 'study'...The phrase '*qua* being', in other words, does not indicate which type or sort of beings are under discussion: it indicates *the way in which* beings are going to be discussed.

In a parenthetical remark, Barnes adds: 'In Aristotle's Greek the situation is a little more complex; but the main point remains'.

Thus to describe the properties that John *qua* father has does not mean to describe the properties of a mysterious entity different from John, but to describe those properties of John which bear on the fact or are relevant to the fact that John is a father. Although Barnes speaks of this point as 'trifling', some modern theoreticians have missed it (see below).

Further evidence for Barnes's main point comes from Aristotle's own writings

> A term which is repeated in the propositions ought to be joined to the first extreme, not to the middle. I mean for example that if a deduction should be made proving that there is knowledge of justice, that it is good, the expression 'that it is good' (or '*qua* good') should be joined to the first term. Let A stand for knowledge that it is good, B for good, C for justice. It is true to predicate A of B. For of the good there is knowledge that it is good. Also it is true to predicate B of C. For justice is identical with a good. In this way, an analysis of the argument can be made. But if the expression 'that it is good' were added to B, there will be no analysis; for A will be true of B, but B will not be true of C. For to predicate of justice the term 'good that it is good' is false and not intelligible. (*Prior Ana.* 49a 11-21)

Although there have been some recent attempts to incorporate the preposition *qua* in logic as those of Fine [4] and Landman [6], they do not appear to conform to Aristotle's intuition. For Landman, John, John qua father and John qua husband are different sets of intensional properties. For Fine, John qua father is a 'new object...some sort of amalgam of the given object and the property...'. At any rate, we believe that the logic of *qua*, in the way that Aristotle understood it, has not survived up to now the passage to modern logic.

2 Analysis of aspects

2.1 An imaginary discussion

Suppose that John's honesty is the subject of a discussion. To the question whether John is honest, somebody may answer, quite intelligibly, 'Yes and no'. If asked the further question 'What do you mean?', he may answer 'It depends: in some aspects John is honest, in others, he fails to be so'. Such a discussion is perfectly understandable. Of course there may be disagreement among the participants on which aspects are considered relevant as well as disagreement on John's honesty or lack of honesty under a particular aspect. However, if agreement have been reached on these points, a global judgement about John's honesty may now be reached, apparently, 'by logic alone'. In this section we show how to conceptualize such an analysis of aspects as a category of 'aspects', in a way that we believe agrees with Aristotle's intuition; in the following section we discuss models which allow us to state 'honesty (or lack of honesty) of John under a given aspect' and to arrive at global judgements of the type 'John is honest', 'John is not honest' and 'John is dishonest' from this information. In this way, we show the intimate connection between the logic of *qua* and predicate term negation. We have found no evidence that Aristotle saw this connection.

Assume that John is a family man (a husband and a father), a businessman and a politician and agreement has been reached that these are the only relevant aspects to arrive at a global judgement on John's honesty: John *qua* family man, businessman and politician (global aspect), John *qua* family man (family aspect), John *qua* politician (political aspect), John *qua* businessman (business aspect), John *qua* father (parental aspect), John *qua* husband (marital aspect), the last two being subaspects of the family aspect.

First, a few words about such an analysis. There is nothing mysterious about this notion of '$A's$ *qua* B' or '$A's$ *as* B', which plays an important rôle in Aristotle's philosophy as we saw above. For instance, it is common to ask a politician whether he is expressing an opinion as the spokesman of his party or as a private individual. It is perfectly understandable for a mother to complain that her husband, although good as a father, is not good as a husband. Notice, however, that in agreement with Aristotle's intuition (and our own), the '$A's$ *qua* B' do not constitute a kind above and beyond the kind of A's. In particular, John *qua* father is not a mysterious entity different from John. The *qua* indicates that questions about John *qua* father must bear on, or must be relevant to the fact that he is a father. Of course, these questions are, afortiori, relevant to the fact that John is a father *and* a husband, and hence they are questions about John *qua* family man. There could be, on the other hand, questions about John *qua* father and husband, for instance relevant to John *qua* father, which are not relevant to the marital aspect of John. These remarks will guide our definition of the category of aspects relevant to the question of John's honesty.

To define such a category, we recall the definition of $\mathcal{CN}$, the *nominal category*, whose objects are 'genuine' CNs relevant to the discussion such as 'a politician', 'a businessman', 'a family man', 'a father', 'a husband', 'a family man, a businessman and a politician' which we will write as:

$$\boxed{\text{a p}}\ ,\ \boxed{\text{a b}}\ ,\ \boxed{\text{a f}}\ ,\ \boxed{\text{a fa}}\ ,\ \boxed{\text{a h}}\ \text{and}\ \boxed{\text{a fbp}}$$

respectively, and whose morphisms are postulates of the form

$$\boxed{\text{a f}} \xrightarrow{is} \boxed{\text{a fa}}\ ,$$
$$\boxed{\text{a fbp}} \xrightarrow{is} \boxed{\text{a f}}.$$

The identity morphisms are particular axioms of the form

$$\boxed{\text{a p}} \xrightarrow{is} \boxed{\text{a p}}$$

and composition is given by Modus Ponens. For instance from

$$\boxed{\text{a fbp}} \xrightarrow{is} \boxed{\text{a f}}\ \text{and}$$
$$\boxed{\text{a f}} \xrightarrow{is} \boxed{\text{a fa}}\ \text{we obtain}$$
$$\boxed{\text{an fbp}} \xrightarrow{is} \boxed{\text{a fa}}.$$

We may think of these axioms as a system of *identifications* which replaces the notion of *equality* between different kinds, equality being a relation that may hold only between members of a given kind. (See [7] and [8] for details).

2.2 The aspectual category

Recall (e.g. [14]) that if $\mathcal{C}$ is an arbitrary category, *the category of arrows of* $\mathcal{C}$, $\mathcal{C}^{\rightarrow}$, is the new category whose *objects* are morphisms $A \longrightarrow B$ of $\mathcal{C}$ and whose *morphisms* from the object $A \longrightarrow B$ to the object $A' \longrightarrow B'$ are couples $(A \longrightarrow A', B \longrightarrow B')$ of morphisms of $\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

commutes. We have the following diagram of functors

$$\mathcal{C}^{\rightarrow} \begin{array}{c} \xrightarrow{\partial_0} \\ \xleftarrow{id} \\ \xrightarrow[\partial_1]{} \end{array} \mathcal{C}$$

where $\partial_0(A \longrightarrow B) = A$, $\partial_1(A \longrightarrow B) = B$ and $id(A) = A \xrightarrow{1_A} A$, the identity morphism.

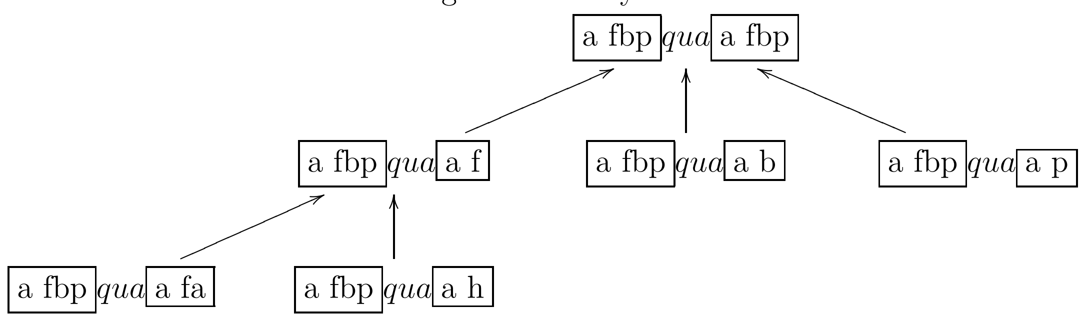
Remark 2.2.1 Since $\partial_i \circ id = Id_{\mathcal{C}}$ for $i = 0, 1$ we conclude that id is an embedding, i.e., a fully faithful functor. Thus, we can identify $A \in |\mathcal{C}|$ with the object $A \xrightarrow{1_A} A \in |\mathcal{C}^{\rightarrow}|$.

Definition 2.2.2 The *aspectual category of* $\mathcal{CN}$*,* $\mathcal{A}(\mathcal{CN})$*,* is defined to be $(\mathcal{CN}^{\rightarrow})^{op}$

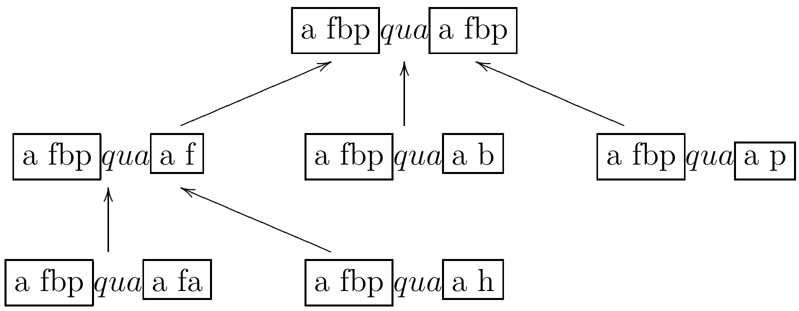
Remark 2.2.3 Since we have assumed that $\mathcal{CN}$ is a pre-ordered set, we can have at most one morphism from an object A to an object B. If such a morphism exists, we let A *qua* $B = A \longrightarrow B$. In particular, A *qua* $A = A \xrightarrow{1_A} A$. By identifying $A \xrightarrow{1_A} A$ with A (as suggested above), the count noun A is identified with its 'global' aspect A *qua* A, i.e., $A = A$ *qua* A.

The following subcategory $\mathcal{A}$ of $\mathcal{A}(\mathcal{CN})$ $(= (\mathcal{CN}^{\rightarrow})^{op})$, the so-called 'comma category' $\boxed{\text{a fbp}} \downarrow \mathcal{CN}$ in [14], will play a fundamental rôle in this paper.

diagrammatically

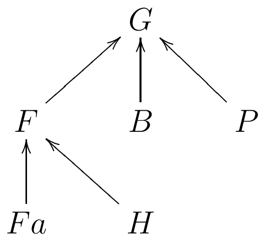


A few words of explanation: neither identity maps nor maps resulting from composition are indicated. The maps that are indicated, on the other hand, are the obvious ones. For instance, $\boxed{\text{a fbp}}$ *qua* $\boxed{\text{a fa}} \longrightarrow \boxed{\text{an fbp}}$ *qua* $\boxed{\text{a f}}$ is given by the couple of morphisms of $\mathcal{CN}$ (going in the opposite direction!)



where $\boxed{\text{a f}} \xrightarrow{is} \boxed{\text{a fa}}$ is the postulate that says that 'a family man is a father', indicating the way in which we understand the expression 'family man'.

Thus, we may view $\mathcal{A}$ as a conceptualization of the aspects of John that were deemed relevant for the discussion of John's honesty: '*qua* fbp', the global aspect G at the top; '*qua* family man', the family aspect F in the second level (leftmost), having itself two subaspects: '*qua* father', the paternal aspect Fa and '*qua* husband', the marital aspect H in the third level, etc. Using these abbreviations, the category $\mathcal{A}$ becomes



which is precisely the analysis of aspects considered in [9]. In that paper, the existence of such a category of aspects, having some semantical properties was postulated. In this paper these properties will be consequences of the way we interpret $\mathcal{A}$ as we will see in the next section.

3 Models

3.1 Interpretations

In this section we will define the notion of interpretation of $(\mathcal{A}, \mathcal{P})$, where $\mathcal{A}$ is the category of 'aspects of John' defined in section 2.2 and $\mathcal{P}$ is a set of predicables which are applicable in a meaningful way to the count nouns of $\mathcal{CN}$.

Our strategy is to define an interpretation of $(\mathcal{A}(\mathcal{CN}), \mathcal{P})$ and then restrict it to one of $(\mathcal{A}, \mathcal{P})$ using the fact that $\mathcal{A}$ is a (full) subcategory of $\mathcal{A}(\mathcal{CN})$.

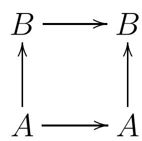
Definition 3.1.1 An *interpretation of* $(\mathcal{A}(\mathcal{CN}), \mathcal{P})$ is a functor

$$X : (\mathcal{A}(\mathcal{CN}))^{op} \longrightarrow Set$$

together with a set $\{X_\phi \hookrightarrow X \mid \phi \in \mathcal{P}\}$ of subfunctors of X satisfying the following conditions*:*

(i) $X(A\ qua\ B) = X(A\ qua\ A)$

(ii) $X_\phi(A\ qua\ B) = X_\phi(B\ qua\ B) \circ X(B\ qua\ B \longrightarrow A\ qua\ A)$, where $B\ qua\ B \longrightarrow A\ qua\ A$ corresponds to the morphism



of $\mathcal{CN}^{\rightarrow}$.

Remark 3.1.2 1. In (ii) we have considered $X_\phi(A\ qua\ B)$ and $X_\phi(B\ qua\ B)$ as maps into $\{\top, \bot\}$, by using characteristic functions. Diagrammatically, we may express (ii) as saying that

$$\begin{array}{ccc} X(A\ qua\ A) & \longrightarrow & X(B\ qua\ B) \\ & {\scriptstyle X_\phi(A\ qua\ B)} \searrow \quad \swarrow {\scriptstyle X_\phi(B\ qua\ B)} & \\ & \{\top, \bot\} & \end{array}$$

is commutative.

2. Since $\mathcal{A}(\mathcal{CN}) = (\mathcal{CN}^{\rightarrow})^{op}$, its dual, $\mathcal{A}(\mathcal{CN})^{op} = \mathcal{CN}^{\rightarrow}$ and so X becomes a covariant functor
$$X : \mathcal{CN}^{\rightarrow} \longrightarrow Set$$

3. By identifying '*A qua B*' with '$X(A\ qua\ B)$' to simplify the notation, this definition says that the $A's$ *qua* B are just the $A's$; but a predicate typed by the aspect A *qua* B is obtained from the corresponding predicate typed by B ($= B\ qua\ B$) by composing it with the identification map $A \xrightarrow{u} B$. In other words, $a \in A$ satisfies $\phi_{A\ qua\ B}$ iff its underlying $u(a)$ in B satisfies ϕ_B, the predicate ϕ typed by B. (Recall that all predicates are typed!). Thus the $A's$ *qua* B do not constitute a new kind over and above A, but the predicates are seen with the eyes of B, as it were. This agrees with the informal discussion of section 2.1. Finally, functoriality says, for instance, that if John is honest qua family man, then he is honest qua father.

Of course nothing guarantees up to now the existence of an interpretation for the aspectual category of CN's and the predicables. This is not so for CN's and predicables for which we have even a standard interpretation: $\boxed{\text{a f}}$ is interpreted as the set of family men, $\boxed{\text{a fbp}}$ is interpreted as the set of family men who are businessmen and politicians, etc. Thus, it is reassuring to have the following

Proposition 3.1.3 *There is a one-to-one correspondence between interpretations of* $(\mathcal{A}(\mathcal{CN}), \mathcal{P})$ *and interpretations of* $(\mathcal{CN}, \mathcal{P})$.

Proof: (Sketch). Recall that we have an embedding $\mathcal{CN} \hookrightarrow \mathcal{A}(\mathcal{CN})$ which allows us the identification of an object A of $\mathcal{CN}$ with the object $A \xrightarrow{1_A} A$ of $\mathcal{A}(\mathcal{CN})$. Clearly, any interpretation of $(\mathcal{A}(\mathcal{CN}), \mathcal{P})$ restricts to one of $(\mathcal{CN}, \mathcal{P})$. Assume that X^o is an interpretation of $(\mathcal{CN}, \mathcal{P})$. We define $X : \mathcal{A}(\mathcal{CN})^{op} \longrightarrow Set$ and $X_\phi \hookrightarrow X$ (for $\phi \in \mathcal{P}$) as follows:

$$X(A \textit{ qua } B) = X^o(A)$$

$$X_\phi(A \textit{ qua } B) = X^o_\phi(B) \circ X^o(A \longrightarrow B)$$

Further details are easy to check.

Remark 3.1.4 Interpretations constitute a category whose morphisms are natural transformations $X \xrightarrow{\eta} Y$ such that their restrictions to $X_\phi \hookrightarrow X$ factor through $Y_\phi \hookrightarrow Y$. This allows us to improve the previous proposition to the statement that the category of interpretations of $(\mathcal{A}(\mathcal{CN}), \mathcal{P})$ is equivalent to the category of interpretations of $(\mathcal{CN}, \mathcal{P})$. This fact, however will not be used in this paper.

Definition 3.1.5 *Let* $\mathcal{B} \overset{i}{\hookrightarrow} \mathcal{A}(\mathcal{CN})$. *An interpretation of* $(\mathcal{B}, \mathcal{P})$ *is the restriction of an interpretation of* $(\mathcal{A}(\mathcal{CN}), \mathcal{P})$.

More precisely, if X is an interpretation of $(\mathcal{A}(\mathcal{CN}), \mathcal{P})$, then the functor

$$X \circ i : \mathcal{B} \longrightarrow Set$$

together with the subfunctors $(X \circ i)_\phi \hookrightarrow X \circ i$ defined by $(X \circ i)_\phi(B) = X_\phi(iB)$ for each $\phi \in \mathcal{P}$, is an interpretation of $(\mathcal{B}, \mathcal{P})$.

Applied to our category $\mathcal{A}$, such an interpretation boils down simply to a set X together with a set of predicates of X where by a *predicate of* X we mean a family $\{\phi_C\}_{C \in |\mathcal{A}|}$ of subsets of X with the following functoriality property: if $x \in \phi_C$ (we write also '$x \in_C \phi$') and $C' \to C \in \mathcal{A}$, then $x \in \phi_{C'}$. In fact an interpretation of $\mathcal{A}$ associates with every aspect the same set X, since all aspects are of the form $\boxed{\text{a fbp}}$ *qua* B. (Recall that $\mathcal{A}$ is the 'comma' category $\boxed{\text{a fbp}} \downarrow \mathcal{CN}$). This is precisely the way we interpreted the category of aspects in our previous publications on non-Boolean negation in natural language.

Let $\mathcal{P}(X)$ be the set of all predicates of X. The predicates form a 2-negation bounded distributive lattice

$$(\mathcal{P}(X), \leq, \vee, \wedge, \top, \bot, \neg, \sim)$$

where
$\phi \leq \psi$ iff $\forall C \in \mathcal{A}\, \forall x \in X\ (x \in_C \phi \Rightarrow x \in_C \psi)$
$x \in_C (\phi \vee \psi)$ iff $x \in_C \phi$ or $x \in_C \psi$
$x \in_C (\phi \wedge \psi)$ iff $x \in_C \phi$ and $x \in_C \psi$
$\bot$ is the predicate 'false', i.e., the bottom element of the order and $\top$ is the predicate 'true', i.e., the top element of the order.

Given a predicate ϕ, we define two new predicates $\neg\phi$, $\sim \phi$ as follows:
$x \in_C \neg\phi$ iff $\forall C' \to C \in \mathcal{A}\ \ x \notin_{C'} \phi$
$x \in_C \sim \phi$ iff $\exists C \to C'\ \in \mathcal{A}\ \ x \notin_{C'} \phi$

Notice that $\neg\phi$ and $\sim \phi$ as defined are indeed predicates, ie, they have the functorial property. Applied to our main example, the predicate '$\neg$ honest' will be read as 'dishonest' and '$\sim$ honest' as 'not honest'. From the definitions of the two negations it is easy to prove the following:

Proposition 3.1.6 *(Adjunction properties of negations.) The following holds for arbitrary properties ϕ and ψ :*

1. $\dfrac{\psi \leq \neg\phi}{\psi \wedge \phi = \bot}$

2. $\dfrac{\sim \phi \leq \psi}{\top = \psi \vee \phi}$

Corollary 3.1.7 *The following holds for every property ϕ:*
$\phi \vee \sim \phi = \top$, $\phi \wedge \neg\phi = \bot$

Notice however that $\phi \wedge \sim \phi$ is not necessarily the false $\bot$. Similarly $\phi \vee \neg\phi$ is not necessarily the true $\top$. Examples of this phenomena will be given in section 3.2

3.2 Global judgement

Let us return to the discussion about John's honesty. Assume that agreement has been reached on which aspects are considered relevant as well as agreement on John's honesty or lack of honesty under each (relevant) aspect. In other words, agreement has been reached, for each aspect C whether $J \in_C h$ or $J \notin_C h$. Then the global judgement is obtained *by restricting to the global aspect* G. This means that John is honest, not honest or dishonest precisely when $J \in_G h$, $J \in_G \sim h$, $J \in_G \neg h$.

Spelling this out in detail and writing '$J \in \phi$' for $J \in_G \phi$, we obtain

> $J \in h$ iff $\forall C\ J \in_C h$ ('John is honest iff John is honest under any aspect')
>
> $J \in\sim h$ iff $\exists C\ J \notin_C h$ ('John is not honest iff John fails to be honest under at least one aspect')
>
> $J \in \neg h$ iff $\forall C\ J \notin_C h$ ('John is dishonest iff John fails to be honest under every aspect').

Let us point out some consequences: it is not always the case that John is either honest or dishonest. Assume, for instance, that $X = \{John\}$ and that John is honest under the aspect B (i.e., John is honest qua businessman), but fails to be honest under every other aspect. On the other hand, if John is honest, he cannot be dishonest and vice versa. This is to be expected, since the negations are not Boolean. It is more surprising, however, that $\sim$ is Boolean *at the global level.* Thus, is is always the case that John is either honest or not honest, but not both. Similarly, it is always the case that John is dishonest or not dishonest, etc. We believe that this feature of this negation may account for the persistent belief that the logic of natural language is Boolean.

All of this is rather straightforward. Nevertheless to interpret some combinations of these negations we have to return to calculations in the model. Here a new feature appears: some of these combinations lack names in natural languages. This does not make them less understandable as we will see. Without trying to be exhaustive, we will compute a few combinations (the horizontal line stands for if and only if):

1. $$\frac{\frac{\frac{J \in_C \sim \neg h}{\exists C \longrightarrow C' \; J \notin_{C'} \neg h}}{\exists C \longrightarrow C' \; \exists C'' \longrightarrow C' \; J \in_{C''} h}}{\exists C' \; J \in_{C'} h}$$

   (The last equivalence holds because any two aspects are connected to G.)

2. $$\frac{\frac{J \in_C \sim\sim h}{\exists C \longrightarrow C' \; J \notin_{C'} \sim h}}{\exists C \longrightarrow C' \; \forall C' \longrightarrow C'' \; J \in_{C''} h}$$

3. $$\frac{\frac{\frac{J \in_C \neg \sim h}{\forall C' \longrightarrow C \; J \notin_{C'} \sim h}}{\forall C' \longrightarrow C \; \forall C' \longrightarrow C'' \; J \in_{C''} h}}{\forall C' \; J \in_{C'} h}$$

4. $$\frac{\frac{J \in_C \neg\neg h}{\forall C' \longrightarrow C \; J \notin_{C'} \neg h}}{\forall C' \longrightarrow C \; \exists C'' \longrightarrow C' \; J \in_{C''} h}$$

Restricting these results to the global level $C = G$, we obtain

1. $$\frac{J \in \sim \neg h}{\exists C \; J \in_C h}$$

   ('John is not dishonest iff John is honest under at least one aspect')

2. $$\frac{J \in \sim\sim h}{J \in h}$$

   ('John is not not honest iff John is honest')

3. $$\frac{J \in \neg \sim h}{J \in h}$$

4. $$\frac{J \in \neg\neg h}{\forall C \; \exists C' \longrightarrow C \; J \in_{C'} h}$$

Some people have objected to our model on the basis that if John is honest under the aspects F, B and P, then it should follow that John is honest, since there are no other aspects to be considered. This need not happen in our model, since we have a further global aspect G. This objection can be taken care of provided that we interpret predicables such as 'honest' as *regular* predicates, i.e., predicates ϕ such that $\neg\neg\phi = \phi$. Thus if 'honest' is interpreted as a regular predicate, then John is honest under an aspect iff John is honest under all aspects immediately below, as can be checked immediately from our computations. Thus John is honest under F iff John is honest under Fa and under H. Similarly, John is honest iff John is honest under the aspects F, B and P. Since regular predicates are in particular predicates, the whole theory applies to them. Furthermore, it is not always the case that John is either honest or dishonest (the same example works here). Let us point out that there are a plethora of such predicates. Indeed, if ϕ is an arbitrary predicate, $\neg\neg\phi$ is a regular predicate, in fact the smallest regular predicate containing ϕ.

A further objection concerns the negation $\sim$: as easily checked, John is not honest under a given aspect iff he is not honest under every aspect. But notice that there is a difference between not honest under an aspect and failing to be honest under that aspect. The first is governed by 'forcing' clauses and has the functoriality property: if John is not honest under the aspect C, then he is not honest under any subaspect $C' \longrightarrow C$. Furthermore, it may even happen that John is both honest and not honest under one and the same aspect (although not the global one). In the example already discussed, John is both honest and not honest under the aspect B. In fact, John fails to be honest under the global aspect, otherwise he would be honest under every aspect (by functoriality). But this implies that John is not honest under the global aspect and thus, by functoriality, that John is not honest under every aspect, in particular B.

The second (failing to be honest under a given aspect), on the other hand, is just the absence of John's honesty under that aspect. John may fail to be honest under an aspect and be honest under another. In fact, absence of John's honesty under that aspect is not even functorial and it is not possible for John to be honest and to fail to be honest under one and the same aspect. We believe that the objection results from confusing these notions. They coincide only for the global aspect and this is the only aspect that 'surfaces' at the natural language level. At the global aspect, however, the logic of $\sim$ is Boolean and all is well.

3.3 Modalities

From the fact that John is not dishonest we cannot conclude that John is honest, but only that he is possibly so. This suggests a definition of a modal operator $\Diamond\phi = \sim \neg\phi$ to be read as 'possibly ϕ'. This interpretation finds further support in the calculation above: John is not dishonest under a given aspect precisely when he is honest under at least one aspect. Similarly, we may define a modal operator $\Box\phi = \neg \sim \phi$ to be read as 'necessarily ϕ'. According to the previous calculation, John is necessarily honest under a given aspect iff he is honest under every aspect. The following is left as an exercise to the reader:

Proposition 3.3.1 *The operators $\Diamond$ and $\Box$ have the following properties (where ϕ is a predicate):*

(1) $\Box\phi \leq \phi \leq \Diamond\phi$

(2) $\Box\Box\phi = \Box\phi$, $\Diamond\Diamond\phi = \Diamond\phi$

(3) $\phi \leq \Box\Diamond\phi$

(4) $\Diamond\Box\phi \leq \phi$

(5) $\Diamond(\phi \wedge \Box\psi) = \Diamond(\phi) \wedge \Box(\psi)$ *(Frobenius law)*

(6) $\Box\phi \vee \neg\Box\phi = \top$

(7) If $\phi \vee \neg\phi = \top$, *then* $\Box\phi = \phi$

There are some unusual features of this modal logic. First it is not Boolean for contingent predicates, although it is Boolean for modally closed ones (i.e., of the form either $\Box\phi$ or $\Diamond\phi$). Furthermore, the only predicates to satisfy the excluded middle are in fact the modally closed ones. Properties (2) and (3) express the adjointness relation: $\Diamond \dashv \Box$. The other properties are rather standard for a modal logic. This modal logic was introduced in [22]. It was proved to be decidable in [20]. On the other hand, completeness of this logic seems to be an open problem. The interested reader is referred to these papers for further information.

3.4 Logic of terms versus logic of propositions

Up to now we have discussed negations of predicates which applied, for instance, to 'honest' yield the new predicates 'not honest' and 'dishonest'. These correspond to the predicate negation and predicate term negation, respectively, in Aristotle's logic of terms, as explained in section 1. We will now show that negations of propositions can also be handled in our context. Thus we achieve a certain unification of term logic and propositional logic.

The most expedient way to achieve the unification is to interpret a sentence as a predicate of a singleton $1 = \{*\}$. In fact a sentence either holds or fails to hold under a given aspect; but if it holds, then it should do so for every subaspect. We will write '$J \in h$' for the sentence 'John is honest'. It seems natural to interpret this sentence as the predicate of * which holds under C iff $J \in_C h$. Using our notation,

$$* \in_C (J \in h) \textit{ iff } J \in_C h$$

Other clauses (corresponding to logical connectives) have already been defined, but we will recall some to compare them with the corresponding clauses of the logic of predicates of X:

$$* \in_C \neg(J \in h) \text{ iff } \forall C' \longrightarrow C \;\; * \notin_{C'} (J \in h)$$

$$* \in_C \sim (J \in h) \text{ iff } \exists C \longrightarrow C' \;\; * \notin_{C'} (J \in h)$$

These clauses allow us to conclude at once:

Proposition 3.4.1 *(Compatibility of term negation with propositional negation)*

$$J \in_C h \;\; \textit{iff} \;\; * \in_C (J \in h)$$

$$J \in_C \neg h \;\; \textit{iff} \;\; * \in_C \neg(J \in h)$$

$$J \in_C \sim h \; \textit{iff} \; * \in_C \sim (J \in h)$$

We will read $\sim \sigma$ as 'it is not the case that σ' or 'it is false that σ ' and $\neg\sigma$ as 'it is utterly false that σ'. Returning to our computations of section 3.2 we can conclude (using this proposition) that 'it is utterly false that John is not honest iff John is honest'. On the other hand, if it is utterly false that John is dishonest, one cannot conclude that John is honest. This may not seem intuitive to some people. Notice, however, that this phenomenon cannot happen if we interpret 'honest' as a regular predicate, i.e., a predicate ϕ such that $\neg\neg\phi = \phi$.

4 Appendix

Those familiar with category theory have noticed that the interpretation X of $(\mathcal{A}, \mathcal{P})$ is an object of the presheaf topos $Set^{\mathcal{A}^{op}}$ together with a set of subobjects corresponding to the predicables of $\mathcal{P}$. The global aspect G is the terminal object of $\mathcal{A}$ and the restriction of X to G is obtained by applying the global sections functor Γ of the geometric morphism (Δ, Γ) where $\Delta \dashv \Gamma$:

$$Set^{\mathcal{A}^{op}} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} Set$$

In the left side of this morphism, we have the rich logic given by the category $\mathcal{A}$ of the analysis of aspects. On the right side, we have the Boolean logic of global judgements.

Recall that an interpretation assigns the same set to all the aspects. In our context this means that the interpretation X is a constant presheaf. Also the fact that $\mathcal{A}$ has a terminal object implies that $\mathcal{A}$ is connected, so $\Gamma\Delta = Id$ and Δ is full and faithful. Thus every morphism between two constant presheaves is a constant morphism.

The global judgement takes place in Set. The functor Γ 'forgets' the aspects of X and simply gives $\Gamma X = X(G)$, the set of 'individuals'. For a predicate ϕ of X, $\Gamma\phi$ is just a set-theoretic predicate (i.e., a subset) of ΓX, and it corresponds to the restriction of ϕ to the global level.

The question we are studying is how the logic of aspects is transformed by Γ. First, let us describe this logic. We know that presheaf toposes are bi-Heyting toposes [12], [22] and thus every X in $Set^{\mathcal{A}^{op}}$ has a bi-Heyting structure $(\mathcal{P}(X), \bot, \top \wedge, \vee, \rightarrow, \neg, \backslash, \sim)$ together with substitutions f^* and quantifiers $\forall_f$ and $\exists_f$ for every $X \xrightarrow{f} Y$ in $Set^{\mathcal{A}^{op}}$. We also know how to interpret possibility ($\Diamond$) and necessity ($\Box$) operators. In fact, there are two ways to do so (see [21] and [22]) but in the context of a connected category

$\mathcal{A}$ and for constant objects, they both give the same operators (see [22]). For ϕ in $\mathcal{P}(X)$, we have

$x \in_C \Diamond\phi$ iff for some aspect $C', x \in_{C'} \phi$

$x \in_C \Box\phi$ iff for all aspects $C', x \in_{C'} \phi$ iff $x \in_G \phi$

Notice that $\Diamond\phi$ and $\Box\phi$ with this definition are clearly predicates of X.

The logic of global judgements, on the other hand, includes the Boolean logic of Set. For X constant in $Set^{\mathcal{A}^{op}}$, the predicates of ΓX have the structure:

$$(\mathcal{P}(\Gamma X), \bot, \top, \wedge, \vee, c)$$

(Here $\sim = \neg = c$ and the operations $\rightarrow$ and $\setminus$ are defined with the help of the complement (c)). We have also substitutions and quantifiers along maps. Necessity and possibility operators are simply the identity.

Now let us look at the transformations of these operators by Γ. Γ always preserves $\bot, \top, \wedge, \vee$. In our context, by proposition [9, page 62] $\Gamma \sim = c$ and therefore Γ preserves $\sim$, as in Set $\sim$ is simply the complement. On the other hand $\neg$ is not preserved: $\Gamma\neg \neq c$. For example a proposition might not hold at the global level without being utterly false. We get then a new operation $\Gamma\neg$ on $\mathcal{P}(\Gamma X)$ which is not Boolean but rather comes as the shadow of an underlying analysis.

Thus $\mathcal{P}(\Gamma X)$ becomes a distributive lattice enriched with two negations:

$$(\mathcal{P}(\Gamma X), \bot, \top, \wedge, \vee, c, \Gamma\neg)$$

where $\Gamma\neg$ is the strong negation whose value is calculated in $Set^{\mathcal{A}^{op}}$. In the same way $\setminus$ is preserved and $\rightarrow$ gives a new operation $\Gamma \rightarrow$.

Substitutions and existential quantifiers are always preserved by Γ, in the sense that for $X \xrightarrow{f} Y$, $\Gamma(f^*) = (\Gamma f)^*$ and $\Gamma(\exists_f) = \exists_{\Gamma f}$. In our context, even universal quantifiers are preserved: $\Gamma(\forall_f) = \forall_{\Gamma f}$.

For modal operators, it is easily seen that $\Gamma\Box = \Gamma$, so $\Box$ is preserved, as $\Box$ in Set is simply the identity. This is not the case for $\Diamond$: for example a proposition might be possibly true, i.e., it holds under some aspect, without being true (at the global level). So $\Gamma\Diamond$ is also a new operation on $\mathcal{P}(\Gamma X)$, coming from the underlying analysis in $Set^{\mathcal{A}^{op}}$.

The following proposition summarizes our previous discussion in a slightly more general setting

Proposition 4.0.2 *Let $\mathcal{C}$ be a category having a terminal object and let $f : X \longrightarrow Y$ be a natural transformation between constant presheaves. Then*

(1) $\forall_f \Box = \Box \forall_f$

(2) $\exists_f \Diamond = \Diamond \exists_f$

(3) $\exists_f \Box = \Box \exists_f$

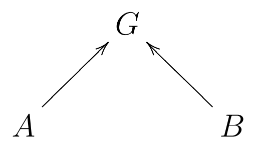
(4) $\forall_f \Diamond = \Diamond \forall_f$

(5) $\sim \exists_f = \forall_f \sim$

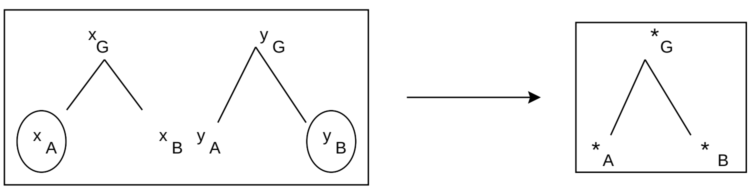
(6) $\sim \forall_f = \exists_f \sim$

(7) $\neg \exists_f = \forall_f \neg$

Notice that although $\exists_f \neg \leq \neg \forall_f$, they are not necessarily equal. For example, let $\mathcal{C}$ be the category:



and let f be the constant $X \xrightarrow{!} 1$



with $\phi = \{x_A, y_B\}$. Here $\forall_! \phi = \bot$, so $\neg \forall_! \phi = \top$, and $*_G \in \neg \forall_! \phi$. On the other hand $x_G \notin \neg \phi$ and $y_G \notin \neg \phi$, therefore $*_G \notin \exists_! \neg \phi$.

Thus, 'it is utterly false that everybody is honest' is not equivalent to 'there is somebody who is dishonest'. On the other hand, (7) says that 'it is utterly false that somebody is honest' is equivalent to 'everybody is dishonest'.

To sum up, we have studied interpretations X which appear either as constant objects in the presheaf topos $Set^{\mathcal{A}^{op}}$ or sets of individuals in Set. We described the interplay between these two points of view. The constant objects of $Set^{\mathcal{A}^{op}}$ have a very rich and regular structure, whereas the logic of Set is Boolean. Through the restriction functor, the Boolean logic is enriched with new operations of strong ($\Gamma\neg$) and possibility ($\Gamma\Diamond$) which are the 'shadows' of their corresponding operations in $Set^{\mathcal{A}^{op}}$, giving the structure

$$(\mathcal{P}(\Gamma X), \bot, \top, \wedge, \vee, c, \Gamma\neg, \Gamma\Diamond).$$

It is this structure that provides a ground for global judgements.

Acknowledgements

John Macnamara and Gonzalo E. Reyes gratefully acknowledge the support of individual grants from the Canada's National Science and Engineering Research Council (NSERC) as well as a collaborative grant from Quebec's Fonds pour la formation de chercheurs et l'aide à la recherche (FCAR). They also acknowledge the support of a collaborative grant from NSERC to B. Gillon, J. Macnamara, M. Makkai and G. E. Reyes. Marie La Palme Reyes is grateful for the support from the collaborative FCAR grant to Macnamara and Reyes. Houman Zolfaghari was partly supported from the NSERC grants of Macnamara and Reyes and from the NSERC collaborative grant.

References

[1] Barnes, J. (ed.) (1984). *The complete Works of Aristotle.* Vol.1. (2nd edition 1985). Bollingen series. Princeton: Princeton University Press.

[2] Barnes, J. (1994). What is Ontology? In V. Gómez Pin (coordinador). *Actas del Primer Congreso Internacional de Ontología.* Bellaterra: Publicacions de la Universitat Autònoma de Barcelona. 35-47.

[3] Englebretsen, G. (1974). A note on contrariety. *Notre Dame Journal of Formal Logic.* Vol.15. 613-614.

[4] Fine, K. (1982). Acts, events and things. In W. Leinfellner, E. Kraemer and J. Schank (eds.). *Language and Ontology: Proceedings of the Sixth International Wittgenstein Symposium.* Wien: Hölder-Pichler-Tempsky.

[5] Horn, L. R. (1989). *A natural history of negation.* Chicago: The University of Chicago Press.

[6] Landman, F. (1989). Groups, II. *Linguistics and Philosophy 12.* Amsterdam: Kluwer Academic Publishers. 723-744

[7] La Palme Reyes, M., J. Macnamara and G. E. Reyes (1994). Reference, kinds and predicates. In J. Macnamara and G. E. Reyes (eds.). *The Logical Foundations of Cognition.* Oxford: Oxford University Press. 91-143.

[8] La Palme Reyes, M., J. Macnamara and G. E. Reyes (1994). Functoriality and grammatical role in syllogisms. *Notre Dame Journal of Formal Logic.* Vol.35, No.1, Winter 1994. 41-66.

[9] La Palme Reyes, M., J. Macnamara, G. E. Reyes and H. Zolfaghari (1994). The non-Boolean logic of natural language negation. *Philosophia Mathematica (3)* Vol.2. 45-68.

[10] La Palme Reyes, M., J. Macnamara, G. E. Reyes and H. Zolfaghari (1994). A category-theoretic approach to Aristotle's logic of terms, with special reference to negation. In V. Gómez Pin (coordinador). *Actas del Primer Congreso Internacional de Ontología.* Bellaterra: Publicacions de la Universitat Autònoma de Barcelona. 241-249.

[11] La Palme Reyes, M., J. Macnamara, G. E. Reyes and H. Zolfaghari (Submitted). Count nouns, mass nouns and their transformations: a unified category-theoretic semantics.

[12] Lawvere, F.W. (1992). Categories of space and of quantity. In J. Echeverría, A. Ibarra and T. Mormann (eds.). *The Space of Mathematics.* Berlin: de Gruyter. 14-30.

[13] Lukasiewicz, J. (1934/1970). On the history of the logic of propositions. In L. Borkowski (ed.) *Jan Lukasiewicz Selected Works.* Amsterdam: North-Holland.

[14] Mac Lane, S. (1971). *Categories for the Working Mathematician.* New York: Springer-Verlag.

[15] Macnamara, J. and G. E. Reyes (eds.) (1994). *The Logical Foundations of Cognition.* Oxford: Oxford University Press.

[16] McCall, S. (1967). Contrariety. *Notre Dame Journal of Formal Logic.* Vol.8. 121-132.

[17] Ockham, William of (1974). *Ockham's Theory of Terms: Part I of the Summa Logicae.* Notre Dame: University of Notre Dame Press.

[18] Ockham, William of (1980). *Ockham's Theory of Propositions: Part II of the Summa Logicae.* Notre Dame: University of Notre Dame Press.

[19] Rescher, N. (1969). *Many-valued logic.* New York: McGraw-Hill.

[20] Reyes, G. E. and M. Zawadowski (1993). Formal systems for modal operators on locales. *Studia Logica.* Vol.52, 4. 595-613.

[21] Reyes, G. E. and H. Zolfaghari (1991). Topos-theoretic approaches to modality. In A. Carboni, M. C. Pedicchio, G. Rosolini (eds.) *Category Theory (Proceedings, Como 1990).* Berlin: Springer-Verlag. 359-378.

[22] Reyes, G. E. and H. Zolfaghari (1996). Bi-Heyting algebras, toposes and modalities. *Journal of Philosophical Logic 25.* 25-43.

[23] Sommers, F. (1970). The calculus of terms. *Mind 79.* 1-39.

[24] Sommers, F. (1982). *The Logic of Natural Language.* Oxford: Clarendon.