Strong Amalgamation, Beck-Chevalley for equivalence relations and Interpolation in Algebraic Logic

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Abstract: We extend Makkai's proof of strong amalgamation (push-outs of monos along arbitrary maps are monos) from the category of Heyting algebras to a class which includes the categories of symmetric bounded distributive lattices, symmetric Heyting algebras, Heyting modal $S4$-algebras, Heyting modal bi-$S4$-albegras, and Łukasiewicz n-valued algebras. We also extend and improve Pitt's proof that strong amalgamation implies Beck-Chevalley for filters of Heyting algebras to exact categories with certain push-outs. As a consequence, a form of the Interpolation Lemma for some non-classical calculi is proved.

Amalgamation properties for certain categories of algebras arising in Algebraic Logic have been used to show Craig's Interpolation Lemma for the corresponding logics. It seems that the first example is due to A. Daigneault who, in the earlier 60's showed that Craig's Interpolation Lemma for classical propositional logic is a consequence of the Amalgamation Property (A) for the category $\mathbb{B}$ of Boolean Algebras (see [3]). More recently, several authors have shown that Craig' Interpolation Lemma for propositional intuitionistic logic (and some extensions) as well as for (classical) propositional S4 logic (and some extensions) are consequences of (A) in the corresponding categories of algebras (see e.g. the exhausting studies of Maksimova in [10], [11], [12] and the references therein indicated).

In 1983, Pitts in his paper [13] introduced a new property that has an independent interest, namely a kind of Beck-Chevalley property for congruence relations in the category $\mathbb{H}$ of Heyting algebras to better conceptualize the proof of Craig's interpolation from (A). (A precise definition will be given in the first section). As a matter of fact, he derived interpolation from a property stronger than (A), Strong Amalgamation (SA).

We say that a category with push-outs has (SA) iff monomorphisms are closed under push-outs, i.e., push-outs of monomorphisms along arbitrary morphisms are monomorphisms. Notice that (SA) implies (A) and hence it is not a good substitute for (A) if we are only interested in interpolation. On the other hand, (SA) is much more definite than (A) in so far as it replaces the existence of some commutative diagram by the monic character of an arrow in a definite diagram (the push-out). Thus, it is interesting to know what categories satisfy (SA).

Pitt's proof has three parts:

(1) (SA) holds in $\mathbb{H}$

(2) (SA) implies the Beck-Chevalley property for congruence relations in $\mathbb{H}$

(3) The Beck-Chevalley property for congruence relations in $\mathbb{H}$ implies Craig's Interpolation Lemma for intuitionistic propositional calculus

In this paper, we shall generalize and improve (2) for any exact category having certain push-outs. Furthermore, we shall extend (1) to several categories of algebras arising from logic such as modal Heyting S4-algebras, modal Heyting bi-S4 algebras (and some variations thereof), symmetric distributive lattices (usually called De Morgan algebras), symmetric Heyting algebras, n-valued Łukasiewicz algebras, etc., by suitably modifying Makkai's proof [8] that the category $\mathbb{H}$ has (SA). Our proof uses the double dual construction for enriched distributive lattices, as developped in our previous paper [4]. (The double dual was also used by Maksimova in the papers already quoted. However, the functorial character of this construction was not stated explicitely in those papers.) Reference [4] and the references therein could be consulted for these algebras and their corresponding logics. Finally, we extend (3) to several of the logics corresponding to categories of algebras in (1) to obtain some versions of the Interpolation Lemma.

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1 Beck-Chevalley for equivalence relations in exact categories

To make this paper reasonably self-contained, we shall review the notion of an exact category. (For further details, see [2]).

We say that a category is *exact* if it has the following properties

(1) It is finitely complete

(2) Every kernel pair has a coequalizer

(3) Regular epimorphisms are stable under pull-backs

(4) Equivalence relations are effective

An epimorphism is *regular* if it is the coequalizer of a pair of parallel arrows.

Recall the following basic facts about regular epimorphisms:

Proposition 1.0.1 *In any category,*

(i) If a regular epimorphism has a kernel pair, then it is the coequalizer of that kernel pair.

(ii) If a kernel pair has a coequalizer, it is the kernel pair of that regular epimorphism

Proof. See [2] (vol I, p.53,54).

Proposition 1.0.2 *If the push-out of a regular epimorphism along a map exists, it is again a regular epimorphism.*

Proof. See [2] (vol I, p.141)

An *equivalence relation* E on A in a finitely complete category is a subobject $E \hookrightarrow A \times A$ such that for every object X the corresponding relation $h^X(E)$ on the set $h^X(A)$ is an (ordinary) equivalence relation (where h^X is the covariant Yoneda functor). An equivalence relation E on A is *effective* if it can be embedded in a coequalizer diagram

$$E \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A \xrightarrow{\tau_E} A/E$$

in such a way that (π_1, π_2) is the kernel pair of q. We let $\mathcal{E}(A)$ to be the poset of equivalence relations on A.

Remark 1.0.3 (i) Exact categories were defined to capture the exactness properties of finite limits and coequalizers of equivalence relations of the category of sets. Barr's metatheorem (cf. [1]) says that this aim is achieved. More precisely, any (small) diagram chasing argument valid in Sets is valid in any exact category, provided the data of the diagram involve only finite limits and coequalizers of equivalence relations. Although vague and hence not susceptible of a 'formal' proof, its truth seems beyond doubt. Of course in particular applications, direct proofs may be provided.

(ii) There are plenty of exact categories: abelian categories, the category of sets, toposes, the category of algebras for an algebraic theory (which includes all the examples dealt with in this paper) and, more generally, monadic categories over Sets. Notice that all these categories are, furthermore, cocomplete, i.e., have initial objects and push-outs. In particular, *push-outs of regular epimorphisms along arbitrary morphisms exist*, a condition that will play a key role in this paper.

(iii) We should point out that this notion of exact category is not the only one around. An exact category is sometimes defined (see e.g. [2]) to be a category with properties (1)', (2), (3), (4), with (1) replaced by the following weaker version:

(1)' Every arrow has a kernel pair

For our purposes, the previous definition is more convenient.

Every map $f : A \longrightarrow B$ in a finitely complete category induces an order-preserving map $f^* : \mathcal{E}(B) \longrightarrow \mathcal{E}(A)$ which associates to an equivalence relation F on B the equivalence relation on A given by the obvious pull-back $(f \times f)^*(F) \hookrightarrow A \times A$.

Proposition 1.0.4 *Let $f : A \longrightarrow B$ be a map in an exact category, $E \in \mathcal{E}(A)$ and $F \in \mathcal{E}(B)$. Then*

(i) The quotient map $\overline{f} : A/E \longrightarrow B/F$ exists (with $\overline{f}\tau_E = \tau_F f$) iff $E \hookrightarrow f^*(F)$

(ii) The quotient map $\overline{f} : A/E \longrightarrow B/F$ exists and is monic iff $E = f^*(F)$

Proof.

(i): We have the following equivalences

$$\begin{array}{c} \exists! \overline{f} : A/E \longrightarrow B/F \text{ with } \overline{f}\tau_E = \tau_F f \\ \hline \tau_F f \pi_1 = \tau_f f \pi_2 \\ \hline \exists! g : E \longrightarrow F \text{ with } \pi_1 g = f\pi_1 \, , \pi_2 g = f \pi_2 \end{array}$$

The first equivalence is a consequence of τ_E being a coequalizer. The second may be proved as follows:

$\downarrow$: Consider the diagram

$$\begin{tikzcd}
& B \arrow[r, "\tau_F"] & B/F \\
& F \arrow[u, "\pi_2"'] \arrow[r, "\pi_1"] & B \arrow[u, "\tau_F"'] \\
E \arrow[uur, "f_{\pi_2}"] \arrow[urr, "f_{\pi_1}"'] & &
\end{tikzcd}$$

The inner square is a pull-back by proposition 1.0.1 while the outer diagram is commutative by hypothesis. By definition of pull-back, $\exists! g : E \longrightarrow F$ such that $\pi_1 g = f\pi_1$ and $\pi_2 g = f\pi_2$

$\uparrow$: Obvious.

Returning to our equivalences, notice that we can rewrite the last as $\exists! g : E \longrightarrow F$ with $(\pi_1, \pi_2)g = (f \times f)(\pi_1, \pi_2)$. But this is clearly equivalent to $E \hookrightarrow f^*(F)$, finishing the proof.

(ii): Use Barr's metatheorem. Since this property plays a key role in our paper, we shall give a direct proof. Assume that $\overline{f}$ is monic. By proposition 1.0.1 the diagram

$$\begin{tikzcd} F \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "\tau_F"] \\ B \arrow[r, "\tau_F"'] & B/F \end{tikzcd}$$

is a pull-back. Equivalently, the diagram

$$\begin{tikzcd} F \arrow[r, "(\pi_1, \pi_2)"] \arrow[d, "(\tau\pi_1, \tau\pi_2)"'] & B \times B \arrow[d, "\tau_F \times \tau_F"] \\ \Delta \arrow[r, "i"'] & B/F \times B/F \end{tikzcd}$$

is a pull-back (where i is the obvious inclusion). By putting on top the diagram which defines $f^*(F)$, we obtain the composite pull-back

$$\begin{tikzcd} f^*(F) \arrow[r] \arrow[d] & A \times A \arrow[d, "\tau_F f \times \tau_F f"] \\ \Delta \arrow[r, "i"'] & B/F \times B/F \end{tikzcd}$$

Once again, we may re-write this diagram as the new pull-back diagram

$$\begin{tikzcd} f^*(F) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "\tau_F f"] \\ A \arrow[r, "\tau_F f"'] & B/F \end{tikzcd}$$

Comparing it with the pull-back

$$
\begin{tikzcd}
E \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "\tau_E"] \\
A \arrow[r, "\tau_E"'] & A/E
\end{tikzcd}
$$

we conclude that $E = f^*F$, since $\overline{f} : A/E \longrightarrow B/F$ is monic.

To show the other direction, consider the diagram

$$
\begin{tikzcd}
E \arrow[r] \arrow[d] & A \times A \arrow[d, "\tau_E \times \tau_E"] \\
\Delta_1 \arrow[r] \arrow[d] & A/E \times A/E \arrow[d, "\overline{f} \times \overline{f}"] \\
\Delta_2 \arrow[r] & B/F \times B/F
\end{tikzcd}
$$

and assume that $E = f^*(F)$, i.e., the outer diagram is a pull-back. So is the top, by properties of exact categories. To show that $\overline{f}$ is monic is clearly equivalent to show the following

Claim: the bottom square is a pull-back.

To prove the claim, let $(\alpha_1, \alpha_2) : X \longrightarrow A/E \times A/E$ and $(\beta, \beta) : X \longrightarrow \Delta_2$ be such that $\overline{f} \times \overline{f} = i_2(\beta, \beta)$ where $i_2 : \Delta_2 \hookrightarrow B/F \times B/F$. Form the pull-back diagram

$$
\begin{tikzcd}
Y \arrow[r, "(a_1, a_2)"] \arrow[d] & A \times A \arrow[d, "\tau_E \times \tau_E"] \\
X \arrow[r, "(\alpha_1, \alpha_2)"'] & A/E \times A/E
\end{tikzcd}
$$

Since $(\overline{f} \times \overline{f})(\tau_E \times \tau_E)(a_1, a_2) = i_2(\beta, \beta)q$ it follows that (a_1, a_2) factors through E by the universal property of the pull-back. By abuse of language, we shall write again $(a_1, a_2) : Y \longrightarrow E$ for this map. We have to show that $(\tau_E \times \tau_E)(a_1, a_2) : Y \longrightarrow \Delta_1$ has a quotient map $X \longrightarrow \Delta_1$. This shows the existence part. Uniqueness is obvious, since i_1 is monic.

The existence of such a quotient is clearly equivalent to the existence of the quotient of the map $\tau_E a_1 (= \tau_E a_2) : Y \longrightarrow A/E$. By proposition 1.0.4, it is enough to show that $ker(q) \hookrightarrow (\tau_E \times \tau_E)^*(\Delta_1)$, where $ker(q)$ is the kernel pair of q. This is a straightforward matter using 'generalized elements' $y_i : Z \longrightarrow Y$ (i=1,2) of Y (defined at a stage Z).

Proposition 1.0.5 *Assume that $\mathbb{A}$ is an exact category such that push-outs of regular epimorphisms along any map exist. If $f : A \longrightarrow B$ is a morphism of $\mathbb{A}$, then the order-preserving map*

$$f^* : \mathcal{E}(B) \longrightarrow \mathcal{E}(A)$$

has a left adjoint

$$\exists_f : \mathcal{E}(A) \longrightarrow \mathcal{E}(B)$$

I.e.,

$$\exists_f \dashv f^*$$

Proof. Consider the push-out

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\tau_E"'] & B \arrow[d, "q"] \\ A/E \arrow[r] & Q \end{tikzcd}$$

From properties of an exact category, more specifically from proposition 1.0.1 and proposition 1.0.2, q is a regular epimorphism and $Q \simeq B/F$ where $F = ker(q)$ is the kernel pair of q. Thus, F is an equivalence relation on B and $B \longrightarrow Q$ is, modulo the above isomorphism, just τ_F. Since the quotient $\overline{f}$ exists by construction, $E \hookrightarrow f^*(F)$. By defining $\exists_f(E) = F$, we obtain

$$E \hookrightarrow f^*(\exists_f(E))$$

Consider now the diagram

$$\begin{tikzcd} & & B/F \\ B \arrow[r] \arrow[urr] & B/\exists_f f^*F \arrow[ur] & \\ A \arrow[r] \arrow[u, "f"] & A/f^*F \arrow[u, "\overline{f}"] \arrow[uur, "\overline{\overline{f}}"'] & \end{tikzcd}$$

The unnamed arrows are the canonical quotient maps and the dotted arrow is the quotient map of 1_B. By proposition 1.0.4, we conclude that

$$\exists_f f^*(F) \hookrightarrow F$$

This completes the proof $\Box$

Remark 1.0.6 For the category of algebras of an algebraic theory, an equivalence relation in the categorical sense is an equivalence relation (in the set-theoretical sense) that is at the same time a subalgebra, i.e., a congruence relation in the usual sense. If E is a congruence relation on an algebra A, A/E is the ordinary quotient algebra and $q : A \longrightarrow A/E$ is the ordinary quotient map. Furthermore, $\exists_f(E)$ has a very simple description: it is the smallest congruence relation on B which contains the relation $\exists_{f\times f}(E)$. i.e., the image of E under $f \times f$. This can be checked directly.

We formulate the *Beck-Chevalley* (BC) property for exact categories such that push-outs of regular epimorphisms along any map exist as follows:

(BC): whenever the diagram

$$\begin{tikzcd} C \arrow[r, "h"] & Q \\ A \arrow[u, "g"] \arrow[r, "f"'] & B \arrow[u, "k"'] \end{tikzcd}$$

is a push-out, the diagram

$$\begin{tikzcd} \mathcal{E}(C) \arrow[r, "\exists_h"] \arrow[d, "g^*"'] & \mathcal{E}(Q) \arrow[d, "k^*"] \\ \mathcal{E}(A) \arrow[r, "\exists_f"'] & \mathcal{E}(B) \end{tikzcd}$$

is commutative

Theorem 1.0.7 *The properties* (BC) *and* (SA) *are equivalent in exact categories having the property that push-outs of regular epimorphisms along any map exist.*

Proof.

(SA) implies (BC): Using the notation of the two previous diagrams, we have to show that

$$k^*\exists_h(G) = \exists_f g^*(G)$$

for any $G \in \mathcal{E}(C)$.

Consider the diagram

$$\begin{tikzcd}
C \arrow[rrr] \arrow[dr] & & & Q \arrow[dl] \\
& C/G \arrow[r, "\overline{h}"] & Q/\exists_h G & \\
& A/g^*G \arrow[u, "\overline{g}"] \arrow[r, "\overline{f}"'] & B/\exists_f g^*G \arrow[u, "\overline{k}"'] & \\
A \arrow[uuu] \arrow[ur] \arrow[rrr, "f"'] & & & B \arrow[uuu] \arrow[ul]
\end{tikzcd}$$

The quotient maps may be shown to exist by a straightforward aplication of proposition 1.0.4 and the fact that $\exists_f \dashv f^*$.

The proof of the above implication is a consequence of the following

Lemma 1.0.8 *The inner diagram is a push-out.*

Proof. Straightforward application of proposition 1.0.4

From proposition 1.0.4 it follows that $\overline{g}$ is monic. From the lemma and (SA), $\overline{k}$ is monic. This implies the conclusion by the same proposition 1.0.4.

(BC) implies (SA): First notice that for any morphism $f : A \longrightarrow B$ in the category.

$$\begin{cases} f \text{ is a monomorphism iff } f^*(\Delta_B) \leq \Delta_A \\ \exists_f(\Delta_A) = \Delta_B \end{cases}$$

where Δ_A is the diagonal (i.e., the discrete equivalence relation on A).

Consider the push-out diagram

$$\begin{tikzcd} C \arrow[r, "h"] & Q \\ A \arrow[u, "g"] \arrow[r, "f"'] & B \arrow[u, "k"'] \end{tikzcd}$$

where g is a monomorphism. By (BC) the diagram

$$\begin{tikzcd} \mathcal{E}(C) \arrow[r, "\exists_h"] \arrow[d, "g^*"'] & \mathcal{E}(Q) \arrow[d, "k^*"] \\ \mathcal{E}(A) \arrow[r, "\exists_f"'] & \mathcal{E}(B) \end{tikzcd}$$

is commutative. In particular,

$$k^*(\Delta_Q) = k^*\exists_h(\Delta_C) = \exists_f g^*(\Delta_C) \leq \exists_f(\Delta_A) = \Delta_B$$

and this shows that k is a monomorphism.

Remark 1.0.9 Condition (BC) for an exact category $\mathbb{C}$ is equivalent to $(BC)^*$, the ordinary (BC) condition for *regular* rather than arbitrary subobjects in the opposite category $\mathbb{C}^0$. This observation gives an alternative way to finish the proof that (SA) implies (BC) which does not use the diagram of quotient maps: indeed, (SA) says that the factorization system epi/regular mono in $\mathbb{C}^0$ is stable (under pull-backs). This clearly implies (BC) for regular subobjects (i.e., formal duals of quotients). Unfortunately, this does not trivialize the proof, since, relative to $\mathbb{C}$, (BC) is a condition on *equivalence relations*, whereas $(BC)^*$ is a condition on quotients. To show the equivalence of the two notions we need to connect equivalence relations (and their inclusions) to quotiens (and their maps) and this is what most the proof is about.

2 Strong Amalgamation for some categories of algebras

In this section we shall extend Makkai's proof of (SA) for the category $\mathbb{H}$ of Heyting algebras (see [8]) to a large class of categories of algebras corresponding to non-classical logics. Unfortunately, we do not have a single result from which all others can be derived, but each case has to be derived anew.

We shall use results from [9] and from our paper [4], where the reader can find un-explained definitions and relevant results. To formulate our results succintly, we shall use the following notation, although it does not always agree with that used elsewhere (for instance in our previous paper [4] we used $\mathbb{DH}$ instead of $\mathbb{SH}$):

(1) $\mathbb{D}$: the category of bounded distributive lattices

(2) $\mathbb{H}$: the category of Heyting algebras

(2)' $\mathbb{B}$: the category of Boolean algebras

(3) co-$\mathbb{H}$: the category of co-Heyting algebras

(4) bi-$\mathbb{H}$: the category of bi-Heyting algebras

(5) $\mathbb{SD}$: the category of symmetric distributive lattices

(6) $\mathbb{SH}$: the category of symmetric Heyting algebras

(7) $\mathbb{IS}4$: the category of modal Heyting $S4$-algebras

(7)' $\mathbb{S}4$: the category of modal $S4$-algebras

(8) bi-$\mathbb{IS}4$: the category of modal Heyting bi-$S4$-algebras

(8)' bi-$\mathbb{S}4$: the category of modal bi-$S4$-algebras

(9) $\mathbb{L}_n$: the category of n-valued Łukasiewicz algebras

A *modal Heyting $S4$-algebra* is a Heyting algebra with an operator $\Box$ satisfying

$$\begin{cases} \Box \leq id \\ \Box^2 = \Box \\ \Box 1 = 1 \\ \Box(a \wedge b) = \Box a \wedge \Box b \end{cases}$$

A *modal Heyting bi-*$S4$*-algebra* is a Heyting algebra with two operators $\Box$ and $\Diamond$ satisfying

$$\begin{cases} \Box \leq id \leq \Diamond \\ \Box^2 = \Box \quad \Diamond^2 = \Diamond \\ \Diamond \dashv \Box \end{cases}$$

Notice that all of these are categories of algebras for an algebraic (i.e. equational) theory and thus are exact, complete and co-complete. In particular, push-outs exist. Furthermore, it is well-known and easy to prove using free algebras that monic maps in a category of algebras for an algebraic theory are just injections in the set-theoretical sense.

Theorem 2.0.10 *All of the previous categories of algebras have* (SA)*.*

Proof. (1) and (2)' are simple applications of the Prime Filter Theorem (PFT), while (3) and (4) appear in [7], athough they are implicit in [8]. Before going into details of the other cases, we shall explain the general scheme of Makkai's proof (cf. [8]) for a category $\mathbb{A}$ of algebras (of an algebraic theory) that are enrichments of $\mathbb{D}$.

Our goal is to show that given a push-out diagram in $\mathbb{A}$

$$\begin{array}{ccc} C & \xrightarrow{f'} & Q \\ g \uparrow & & \uparrow g' \\ A & \xrightarrow[f]{} & B \end{array}$$

such that f is monic, then so is f'.

The first step is to form the push-out in $\mathbb{D}$ (by forgetting the extra structure)

$$\begin{array}{ccc} C & \xrightarrow{h} & P \\ g \uparrow & & \uparrow k \\ A & \xrightarrow[f]{} & B \end{array}$$

Then look at the double dual of P in $\mathbb{D}$ and the evaluation map

$$e_P : P \longrightarrow P^{*+}$$

Introduce an $\mathbb{A}$-structure on P^{*+} in such a way that both $e_p h$ and $e_p k$ are $\mathbb{A}$-morphisms (see below) and consider the diagram

$$\begin{tikzcd} C \arrow[r, "e_P h"] & P^{*+} \\ A \arrow[u, "g"] \arrow[r, "f"'] & B \arrow[u, "e_P k"'] \end{tikzcd}$$

This is obviously commutative and in $\mathbb{A}$. By the universal property of the push-out there is a unique $\alpha : Q \longrightarrow P^{*+}$ such that $\alpha f' = e_P h$ and $\alpha g' = e_P k$.

Now assume that f is monic. By (1) so is h and this implies that $e_P h$ is again monic, since e_P is. But the equation $\alpha f' = e_P h$ shows at once that f' is monic. This concludes the proof.

To introduce the structure on P^{*+} with the required properties, we observe that in practically all the examples an $\mathbb{A}$-structure on a distributive lattice D gives rise to a further 'dual' structure on the poset D^*. Furthermore, a poset P with such a dual structure gives rise to an $\mathbb{A}$-structure on P^+. This 'duality' extends to morphisms. Thus, the double dual inherits the $\mathbb{A}$-structure. Furthermore, e_D preserves this structure.

First, introduce a 'dual' structure on P^* by considering the pull-back diagram (obtained from taking prime filters) and using the 'dual' structures on B^* and C^*

$$\begin{tikzcd} C^* \arrow[d, "g^*"'] & P^* \arrow[l, "h^*"'] \arrow[d, "k^*"] \\ A^* & B^* \arrow[l, "f^*"] \end{tikzcd}$$

Since pull-backs are constructed in the category of posets $\mathbb{P}$ just as in Sets, we have

$$\begin{cases} P^* = \{(\beta, \gamma) : f^*\beta = g^*\gamma\} \\ k^* = \pi_1 \\ h^* = \pi_2 \end{cases}$$

where π_1 and π_2 are the obvious (restictions) of the projections.

Next consider the diagram

$$\begin{tikzcd} C \arrow[r, "h"] & P \arrow[r, "e_P"] & P^{*+} \\ A \arrow[u, "g"] \arrow[r, "f"'] & B \arrow[u, "k"'] \arrow[r, "e_B"'] & B^{*+} \arrow[u, "k^{*+}"'] \end{tikzcd}$$

Since B^{*+} is again an $\mathbb{A}$-algebra, and e_B and k^{*+} are morphisms in this category, $e_P k$ (= $k^{*+} e_B$) is again a morphism in $\mathbb{A}$. A similar argument with C^{*+} shows that $e_P h$ is also a morphism of $\mathbb{A}$. Now for the details.

(5): The further 'dual' structure on $D \in \mathbb{SD}$ is the Birula-Rasiowa's transformation $g_D : D^* \longrightarrow D^*$ defined by $x \in g_D(p)$ iff $x' \notin p$. This map is an anti-isomorphism of period 2. Let g_A , g_B , g_C be the Birula-Rasiowa transformations in the duals of A , B , C respectively and define

$$g_P : P^* \longrightarrow P^*$$

as the product $g_B \times g_C$ restricted to P^*. It is a simple matter to check that P^* is closed under g_P and that g_P is an anti-isomorphism of period 2.

Following [4], we define a De Morgan operation on the double dual P^{*+} as follows:

$$p \in N(X) \text{ iff } g_P(p) \notin X$$

It is straightforward to check that N is indeed a De Morgan negation, i.e., it satisfies

(i) $N(X \cup Y) = N(X) \cap N(Y)$

(ii) $N(X \cap Y) = N(X) \cup N(Y)$

(iii) $NN(X) = X$

From [4] we know that B^{*+} is again in $\mathbb{SD}$ and e_B, , k^{*+} , h^{*+} are also morphisms in this category, concluding the proof.

(6): similar to (5) and left to the reader.

(7), (7)': Following [9] we represent a modal Heyting S4-algebra $(A, \wedge, \vee, \longrightarrow, \Box_A, 0, 1)$ as a morphism $i_A : \Box A \hookrightarrow A$ having a right adjoint r_A, by splitting the idempotent $\Box$. Thus, $\Box A = fix(\Box)$ is the sublattice of fix points of $\Box$ and $\Box_A = i_A r_A$. A $S4$-morphism $f : A \longrightarrow B$ is, under this representation, a boolean map whose restriction f_0 to $\Box A$ lands in $\Box B$. Thus f_0 is a lattice morphism.

Take (as before) the push-out diagram in $\mathbb{D}$

$$\begin{tikzcd} C \arrow[r, "h"] & P \\ A \arrow[u, "g"] \arrow[r, "f"'] & B \arrow[u, "k"'] \end{tikzcd}$$

Furthermore, take the push-out diagram (also in $\mathbb{D}$)

$$\begin{tikzcd} \Box C \arrow[r, "h_1"] & Q \\ \Box A \arrow[u, "g_0"] \arrow[r, "f_0"'] & \Box B \arrow[u, "k_1"'] \end{tikzcd}$$

where f_0 and g_0 are the restrictions of f and g, respectively.

By the universal property of the pull-back, there is a unique morphism $j : Q \longrightarrow P$ in $\mathbb{D}$ such that $jh_1 = hi_C$ and $jk_1 = ki_B$. Although we cannot say too much more about j, we have an explicit description of its dual $j^* : P^* \longrightarrow Q^*$. Indeed, recalling that pull-backs in $\mathbb{P}$ are constructed just as in Sets,

$$\begin{cases} P^* = \{(\beta, \gamma) : f^*\beta = g^*\gamma\} \\ Q^* = \{(\beta_0, \gamma_0) : f_0^*\beta_0 = g_0^*\gamma_0\} \\ j^*(\beta, \gamma) = (i_B^*(\beta), i_C^*(\gamma)) \\ k^* = \pi_1 \ , \ k_1^* = \pi_1 \\ h^* = \pi_2 \ , \ h_1^* = \pi_2 \end{cases}$$

where π_1 and π_2 are the obvious (restictions of the) projections. Notice that $j^*(\beta, \gamma)$ is in fact in Q^* : $f_0^* i_B^*(\beta) = i_A^* f^*(\beta) = i_A g^*(\gamma) = g_0^* i_C^*(\gamma)$.

To define a $\Box$ operation in the double dual P^{*+}, we recall (cf [9]) that j^{*+} has a right adjoint R (as well as a left that will not concern us). We put $\Box = j^{*+}R$ and we readily check that $(P^{*+}, \Box)$ is indeed a $S4$-algebra. Define $u = e_P k$ and $v = e_Q k_1$. To check that u is a morphism of $S4$-algebras it is enough to show that $Ru = vr_B$ as an elementary argument shows. Since $vr_B \leq Ru$ is easily proved, we only need to show that $Ru \leq vr_B$.

We start the proof by noticing the following equivalences

$$\frac{\frac{\frac{\frac{R(u(b)) \subseteq v(r_B(b))}{R(e_P(k(b))) \subseteq e_Q(k_1(r_B(b)))}}{\forall q[q \in R(e_P(k(b))) \Rightarrow q \in e_Q(k_1(r_B(b)))]}}{\forall q[\{\forall p(q \leq j^*(p) \Rightarrow k^*(p) \in e_B(b))\} \Rightarrow r_B(b) \in k_i^*(q)]}}{\forall q[\{\forall p(q \leq j^*(p) \Rightarrow b \in k^*(p))\} \Rightarrow r_B(b) \in k_i^*(q)]}$$

The third equivalence uses the explicit description of the right adjoint R of j^{*+} in [9]

Letting $p = (\beta, \gamma)$ (with $f^*(\beta) = g^*(\gamma)$) and $q = (\beta_0, \gamma_0)$ (with $f_0^*(\beta_0) = g_0^*(\gamma_0)$) we are reduced to prove that

(1) implies (2)

where

(1) $\forall (\beta, \gamma) \in P^*[(\beta_0, \gamma_0) \leq (i_B^* \beta, i_C^* \gamma) \Rightarrow b \in \beta]$

(2) $r_B(b) \in \beta_0$

We proceed by contraposition, using that $k^* = \pi_1$, $k_1^* = \pi_1$, etc.

Assume that $r_B(b) \notin \beta_0$. We need to show the existence of a couple (β, γ) of prime filters in B and C such that

$$f^*(\beta) = g^*(\gamma) \ , \ \beta_0 \subseteq i_B^*(\beta) \ , \ \gamma_0 \subseteq i_C^*(\gamma) \ , \ b \notin \beta$$

We first find β such that $\beta_0 \subseteq i_B^*(\beta)$, $b \notin \beta$, by using the Prime Filter Theorem (PFT).

In fact, let F be the filter generated by $\{i_B(y) : y \in \beta_0\}$ and let I be the ideal generated by b, i.e., $I = \downarrow b$.

We claim that $F \cap I = \emptyset$

Let $y' \in F \cap I$. Since the generating set for F is closed under intersections, this means that $i_B(y) \leq y' \leq b$ for some $y \in \beta_0$. Thus $i_B(y) \leq b$ and this implies that $r_B i_B(y) \leq r_B(b)$. On the other hand, $y \leq r_B i_B(y)$ by adjointness. Thus $y \leq r_B(b)$. But this is a contradiction, since $y \in \beta_0$ but $r_B(b) \notin \beta_0$.

Using PFT, there is a prime filter $\beta \supseteq F$ such that $\beta \cap I = \emptyset$. This prime filter has the required properties.

We now find γ. We first define F to be the filter generated by the set $\{i_C(z) : z \in \gamma_0\} \cup \{g(x) : x \in f^*\beta\}$ and let I be the ideal generated by $\{g(x) : x \notin f^*\beta\}$.

We claim that $F \cap I = \emptyset$

Indeed, let $c \in F \cap I$. It is easily checked that this implies the inequalies $i_C(z) \wedge g(x) \leq c \leq g(x')$ with $x \in f^*\beta$, $z \in \gamma_0$, $x' \notin f^*\beta$. Thus

$$i_C(z) \wedge g(x) \leq g(x')$$

We have the following equivalences

$$\frac{\dfrac{\dfrac{i_C(z) \wedge g(x) \leq g(x')}{i_C(z) \leq g(x) \longrightarrow g(x')}}{i_C(z) \leq g(x \longrightarrow x')}}{i_C(z) \leq g(x'')}$$

where $x'' = x \longrightarrow x'$

Notice that $x'' \notin f^*\beta$ i.e., $f(x'') \notin \beta$, since $x \wedge x'' \leq x'$

The last equivalence implies

$$r_C i_C(z) \leq r_C g(x'') (= g_0 r_A(x''))$$

On the other hand, by adjointness, $z \leq r_C i_C(z)$ and therefore $g_0 r_A(x'') \in \gamma_0$. In turn, this is equivalent to

$$r_A(x'') \in g_0^*(\gamma_0) = f_0^*(\beta_0)$$

But this is obviously equivalent to $f_0 r_A(x'') \in \beta_0$ i.e., $r_B f(x'') \in \beta_0$. Since $\beta_0 \subseteq i_B^*(\beta)$ we conclude that $i_B r_B f(x'') \in \beta$. By adjointness, $i_B r_B f(x'') \leq f(x'')$ and this implies that $f(x'') \in \beta$, a contradiction.

By PFT once again, there is $\gamma \supseteq F$ such that $\gamma \cap I = \emptyset$. The couple (β, γ) satisfies all the required properties, concluding the proof $\Box$

(8), (8)': Just as the preceeding one, defining $\diamond = j^{*+}L$, where L is the left adjoint of j^{*+}. Further details are left to the reader.

(9): This proof has a twist, since wo do not introduce a Łukasiewicz structure directly on the double dual, but on a covering.

Let (D, n_D, r_i) (with $1 \leq i \leq n$) be an L_n-algebra. Its 'dual' structure on D^* consists of the Birula-Rasiowa transformation g_D and the maps $r_i^* : P^* \longrightarrow P^*$ defined by $r_i^* = r_i^{-1}$.

Let (A, n_A, r_i), (B, n_B, s_i), (C, n_C, t_i) be L_n-algebras, where n is the De Morgan negation. Going over to the pull-back diagram obtained by taking prime filters, we define $g : P^* \longrightarrow P^*$ by means of the formula $g = g_B \times g_C$ and $\rho_i = s_i^* \times t_i^*$, for $i = 1, 2, \ldots n$. Both are endofunctions on P^*. We define $R = \bigcup_{1 \leq i \leq n} \rho_i(P^*)$ and we let $j : R \hookrightarrow P^*$ be the obvious inclusion. It is easy to verify from the properties of the $g's$, the s_i and the t_i (cf. [4]) that g and ρ_i are endofunctions of R^* which satisfy the axioms for the dual structure of an L_n-algebra (see [4]). By results in loc.cit. the dual R^+ is an L_n-algebra and $j^+ : P^{*+} \longrightarrow R^+$ is a morphism of L_n-algebras. Furthermore, k^{*+} and e_B are morphisms of L_n-algebras (since, as noticed already, k^* is a projection) and hence it preserves the 'dual' structure.

Thus,

$$\begin{tikzcd} C \arrow[r, "j^+ e_P h"] & R^+ \\ A \arrow[u, "g"] \arrow[r, "f"'] & B \arrow[u, "j^+ e_P k"'] \end{tikzcd}$$

is a commutative diagram in $\mathbb{L}_n$. By the universal property of the pull-back there is a unique map $\gamma : Q' \longrightarrow R^+$ such that $\gamma f' = j^+ e_P h$ and $\gamma g' = j^+ e_P k$.

Claim: If f is mono, then so is $j^+ e_P h$

Notice that this implies (by the first equality) that f' is monic, completing the proof of (9).

To prove the claim, observe that we have the following equivalences

$$\frac{\frac{\frac{\frac{j^{+}e_{p}h(c_{1}) = j^{+}e_{P}h(c_{2})}{\forall r \in R[r \in j^{+}e_{P}h(c_{1})) \text{ iff } r \in j^{+}e_{P}h(c_{2}))}}{\forall r \in R[j(r) \in e_{P}h(c_{1}) \text{ iff } j(r) \in e_{P}h(c_{2})]}}{\forall r \in R[h(c_{1}) \in j(r) \text{ iff } h(c_{2}) \in j(r)]}}{\forall r \in R[c_{1} \in h^{*}j(r) \text{ iff } c_{2} \in h^{*}j(r)]}$$

By definition of R, its elements are of the form $r = \rho_i(\beta, \gamma) = (s_i^*(\beta), t_i^*(\gamma))$, for some $1 \leq i \leq n$. This allows us to rewrite the last equivalence as

$$(1)\forall i\forall\beta\forall\gamma[f^*(\beta) = g^*(\gamma) \Rightarrow \{c_1 \in h^*(s_i^*(\beta), t_i^*(\gamma)) \text{ iff } c_1 \in h^*(s_i^*(\beta), t_i^*(\gamma))\}]$$

The map j^+e_Ph is mono iff $(1) \Rightarrow c_1 = c_2$.

Let us prove the contrapositive. Assume that $c_1 \neq c_2$. By the properties of L_n-algebras, there is some $1 \leq i \leq n$ such that $t_i(c_1) \neq t_i(c_2)$. Without loss of generality, we may assume that $t_i(c_1) \not\leq t_i(c_2)$ and this implies the existence of some i and some $\gamma \in C^*$ such that $t_i(c_1) \in \gamma$ and $t_i(c_2) \notin \gamma$.

We must now find a $\beta \in B^*$ satisfying the condition that $f^*(\beta) = g^*(\gamma)$.

This is a simple application of PFT: let F be the filter generated by $\{f(a) : a \in g^*(\gamma)\}$ and let I be the ideal generated by $\{f(a) : a \notin g^*(\gamma)\}$. If $F \cap I$ were not empty, we would have $f(a) \leq f(a')$ with $a \in g^*(\gamma)$ and $a' \notin g^*(\gamma)$. Since f is monic, $a \leq a'$ and this implies that $a' \in g^*(\gamma)$, a contradiction. By PFT, there is a $\beta \supseteq F$ such that $\beta \cap I = \emptyset$. We have shown that (1) is not true □

Remark 2.0.11 *(i)* For (7)' and (8)' these results may be proved in a simple manner, using a duality between topoboolean algebras and posets with open maps. The interested reader may consult [12] for an exhaustive treatment of classes of extensions of (Boolean) S4 and bi-S4 algebras for which usual amalgamation holds. Unfortunately, Maksimova's proof does not seem to apply, without further arguments, to either (7) or (8). The same seems true for other results that are provable by our methods, such as (SA) for the category of co-Heyting (or bi-Heyting) modal S4-logic (or bi-S4), etc. These we leave to the reader.

*(ii)*We don't know whether (SA) holds for any of the following categories of algebras: Kleene algebras, Nelson algebras, MAO modal algebras (cf. [4]

and [14] for definitions and further information). Furthermore, the main open question is whether there is a general duality theory to encompass the particular examples dealt with in this section. In this vein, we should mention the papers by M. Gehrke and B. Jónsson [5] and [6] which also deal with the double dual functor and adress similar problems.

3 Interpolation

In this last section we generalize Pitts' proof of the interpolation property ([13]) from the category of Heyting algebras to various categories of algebras. As a consequence, we derive a form of the interpolation lemma for several non-classical propositional calculi. These are logical counterparts of some of our categories in section 2 in the same way that intuitionistic propositional calculus is the logical counterpart of $\mathbb{H}$. The connection between calculi and categories of algebras is given by the Lindembaum-Tarski construction.

The scheme of Pitts' proof is the following: first identify the congruences with certain filters and the operation f^* and $\exists_f$ on congruences with corresponding operations on filters. Then apply (BC) to filters generated by one element.

For some of our categories in the list of the beginning of section 2 we define a corresponding notion of filter as follows:

(2) and (2)': An $\mathbb{H}$-filter is just a filter

(6) : An $\mathbb{SA}$-filter is a filter closed under the operation ν defined by $\nu(x) = \neg x'$ where $\neg$ and $()'$ are the Heyting and the De Morgan negation, respectively

(7) and (7)': An $\mathbb{S}4$-filter is a filter closed under $\Box$

(9) : A $\mathbb{L}_n$-filter is a filter closed under s_1 (the first operation of the sequence defining an L_n-algebra)

(Notice that we have used the corresponding numbers to facilitate cross-reference. We shall use $\Phi(A)$ for the poset of filters on A of the corresponding kind)

Proposition 3.0.12 *Given a map $f : A \longrightarrow B$ in one of the corresponding categories, the inverse image f^* sends filters on B into filters on A. Furthermore the map*

$$f^* : \Phi(B) \longrightarrow \Phi(A)$$

has a left adjoint

$$\exists_f : \Phi(A) \longrightarrow \Phi(B)$$

Proof. Straightforward computation. The left adjoint may be described quite simply: $\exists_f(F) =$ the smallest congruence relation on B containing the set theoretical image of F under f.

The main result on the road to interpolation is the following

Proposition 3.0.13 *For each of the notions above we have a bijection between the filters and corresponding congruence relations. This bijection is order-preserving. Furthermore, under this bijection the operations f^* and $\exists_f$ for filters are sent into the corresponding ones for congruence relations.*

Proof. Straightforward computation. We associate with a congruence R on A, the set $\{a : (a, 1) \in R\}$ and check that is a filter (of the appropiate type). Conversely, given a filter F on A define the relation $\{(a, b) \in A \times A : \exists x \in F(x \wedge a = x \wedge b)\}$ on $A \times A$ and check that is a congruence relation (again of the appropiate type). Thus we obtain an association, for every algebra A

$$\phi_A : \mathbb{C}(A) \longrightarrow \Phi(A)$$

where $\mathbb{C}(A)$ is the poset of congruences and $\Phi(A)$ is the set of filters (of the appropiate kind). This association is bijective provided that the algebra is a Heyting algebra, possibly with further structure. This includes most of our examples. Indeed, starting from a filter F we need to show that

$$F = \{a \in A : \exists x \in F(a \wedge x = 1 \wedge x)\}$$

which is obvious.

Starting with a congruence R we need to show that

$$R = \{(a, b) \in A \times A : \exists x \in A\ [(x, 1) \in R \ \text{ and } \ (a \wedge x = b \wedge x)]\}$$

$\Leftarrow$: Since $(x, 1) \in R$ and $(a, a) \in R$, $(a \wedge x, a) \in R$. Similarly $(b \wedge x, b) \in R$. Since $a \wedge x = b \wedge x$, it follows that $(a, b) \in R$.

$\Rightarrow$: Since $(a,b) \in R$ and $(b,b) \in R$, $(a \longrightarrow b, b \longrightarrow b) \in R$, i.e. $(a \longrightarrow b, 1) \in R$. Similarly, $(b \longrightarrow a, 1) \in R$ and this implies that $(x, 1) \in R$, where $x = (a \longrightarrow b) \wedge (b \longrightarrow a)$. Clearly $a \wedge x = b \wedge x$.

The assertion about f^* means precisely that the diagram

$$\begin{tikzcd}
\mathbb{C}(A) \arrow[r, "\phi_A"] & \Phi(A) \\
\mathbb{C}(B) \arrow[u, "f^*"] \arrow[r, "\phi_B"'] & \Phi(B) \arrow[u, "f^*"']
\end{tikzcd}$$

is commutative, while the assertion about $\exists_f$ says that the diagram

$$\begin{tikzcd}
\mathbb{C}(A) \arrow[d, "\exists_f"'] & \Phi(A) \arrow[l, "{\phi_A}^{-1}"'] \arrow[d, "\exists_f"] \\
\mathbb{C}(B) & \Phi(B) \arrow[l, "{\phi_B}^{-1}"]
\end{tikzcd}$$

is commutative.

Take an arbitrary $S \in \mathbb{C}(B)$. The commutativity of the first diagram may be stated as

$$\{a : (a,1) \in (f \times f)^*(S)\} = f^*(\{b : (b,1) \in S\})$$

But this is clear since we have the obvious equivalences

$$\frac{\dfrac{(a,1) \in (f \times f)^*(S)}{(f(a), f(1)) \in S}}{(f(a), 1) \in S}$$

The commutativity of the second follows from that of the first by taking left adjoints.

From a 'practical' point of view, this proposition means that we can work either with congruences or with filters to state (BC). We will work with filters to prove interpolation.

Theorem 3.0.14 (Interpolation Property) *Assume that we have a push-out diagram*

$$\begin{tikzcd} C \arrow[r, "h"] & Q \\ A \arrow[u, "g"] \arrow[r, "f"'] & B \arrow[u, "k"'] \end{tikzcd}$$

in any of the categories below. Then the following holds in each of these categories with $b \in B$ *and* $c \in C$ *:*

$\mathbb{H}$, $\mathbb{B}$*: If* $k(b) \geq h(c)$*, then* $\exists a \in A\ (b \geq f(a)$ *and* $g(a) \geq c)$

$\mathbb{SH}$ *: If* $k(b) \geq t^p(h(c))$ *then* $\exists q \in \mathbb{N}\ \exists a\ (b \geq f(a)$ *and* $g(a) \geq t^q(c))$

$\mathbb{IS}4$, $\mathbb{S}4$*: If* $k(b) \geq h(c)$*, then* $\exists a \in A\ (b \geq f(a)$ *and* $g(a) \geq \Box(c))$

$\mathbb{L}_n$ *: If* $k(b) \geq h(c)$*, then* $\exists a \in A\ (b \geq f(a)$ *and* $g(a) \geq s_1(c))$

where $\nu(x) = \neg x'$*,* $t(x) = x \wedge \nu(x)$ *and the iterations* t^n *are defined as usual starting with* $t^0 = id$*, the identity map.*

Proof. We apply (BC) to filters generated by one element. The question is to characterize these filters in each case, their inverse images and their direct images. Let $f : A \longrightarrow B$ be a morphism (in the corresponding category), $< a >$ the filter generated by $a \in A$ and $< b >$ the filter generated by $b \in B$.

Lemma 3.0.15 *The following holds in each of the cases below*

$\mathbb{H}$, $\mathbb{B}$: $< a >= \uparrow a$

$\mathbb{SH}$: $< a >= \{x : x \geq t^p(a)$ *for some* $p \in \mathbb{N}\}$

$\mathbb{IS}4$ *and* $\mathbb{S}4$: $< a >= \uparrow \Box a$

$\mathbb{L}_n$: $< a >= \uparrow s_1(a)$

Furthermore, in each case

$$\begin{cases} f^*(G) = f^{-1}(G) \\ \exists_f(F) = \{y : \exists x \in F(y \geq f(x))\}. \end{cases}$$

Proof. The assertion about $f^*(G)$ has been already dealt with. Since $\exists_f(F)$ is the smallest filter containing the set theoretical image of F under f, all one has to do is to check that if F is a filter of a given type, then $\exists_f(F)$ is again of the same type. This is a straightforward computation.

For the rest, the only non-trivial case is $\mathbb{SH}$.

The key to the proof of the lemma is the following

Lemma 3.0.16 *The operations ν and t have the following properties*

(i) They preserve $\wedge$. In particular they are order-preserving

(ii) $\forall n \in \mathbb{N}\ (\nu^{n+2}(a) \leq \nu^n(a))$

(iii) $\forall n \in \mathbb{N}\ (\nu^n(t(a)) = t^{n+1}(a))$

Proof. The iterations ν^n are defined as usual starting with $\nu^0 = id$, the identity map.

(i) is clear and (ii) follows from $\nu^2(x) \leq x$, as easily checked. The proof of (iii) proceeds by induction. For $n = 0$ this is obvious. Assume this is true for n. From the induction hypothesis, $t^{n+1}(a) = t(\nu^n(t(a)))$. But

$$\begin{array}{rcl} t(\nu^n(t(a))) & = & \nu^n(t(a)) \wedge \nu^{n+1}(t(a)) \\ & = & \nu^n(a) \wedge \nu^{n+1}(a) \wedge \nu^{n+1}(a) \wedge \nu^{n+2}(a) \\ & = & \nu^{n+1}(a) \wedge \nu^{n+1}(a) \wedge \nu^{n+2}(a) \\ & = & \nu^{n+1}(t(a)) \end{array}$$

The proof of the theorem from the lemma is a simple computation. As an example, let us do $\mathbb{SH}$. From (BC) we know that

$$k^*\exists_h(< c >) = \exists_f g^*(< c >)$$

We have the following equivalences

$$\begin{array}{c} b \in k^*\exists_h(< c >) \\ \hline k(b) \in \exists_h(< c >) \\ \hline \exists p \in \mathbb{N}\ (k(b) \geq h(t^p(c))) \\ \hline \exists p \in \mathbb{N}\ (k(b) \geq t^p(h(c))) \end{array}$$

The last equivalence is a consequence of the fact that h preserves ν and t. Similarly, working with the set of the right,

$$\frac{\dfrac{\dfrac{b \in \exists_f g^*(< c >)}{\exists a \in g^*(< c >)(b \geq f(a))}}{\exists a\ (g(a) \in < c > \ \text{ and } \ b \geq f(a))}}{\exists q \in \mathbb{N}\ \exists a\ (g(a) \geq t^q(c) \ \text{ and } \ b \geq f(a))}$$

This gives interpolation for $\mathbb{SH}$.

To obtain the usual Interpolation Lemma for the corresponding calculi we introduce some notation.

We let $\mathcal{L}(\mathbb{A})$ be the logic corresponding to $\mathbb{A}$, where $\mathbb{A}$ is one of the categories $\mathbb{H}$, $\mathbb{B}$, $\mathbb{SH}$, $\mathbb{IS}4$, $\mathbb{S}4$, $\mathbb{L}_n$. If ϕ is a formula of the language of $\mathcal{L}(\mathbb{A})$, we use $\vdash \phi$ to indicate that ϕ is a theorem of that logic. Finally, if ϕ is a formula in the language of $\mathcal{L}(\mathbb{A})$, we let L_ϕ to be the set of formulas (in the same language) whose propositional variables are contained in those of ϕ. Thus $\mathcal{L}(\mathbb{H})$ is the intuitionistic propositional calculus, $\mathcal{L}(\mathbb{B})$ the classical propositional calculus, $\mathcal{L}(\mathbb{SH})$ the modal symmetric propositional calculus, $\mathcal{L}(\mathbb{IS}4)$ is the intuitionistic $S4$-modal logic, $\mathcal{L}(\mathbb{S}4)$ is the classical $S4$-modal logic and $\mathcal{L}(\mathbb{L}_n)$ is the n-valued Łukasiewicz calculus.

Corollary 3.0.17 (Interpolation Lemma) *Assume that ϕ and ψ are formulas of the language of the corresponding logic. Then the following holds in each of the cases indicated:*

$\mathbb{H}$, $\mathbb{B}$: *Assume that $\vdash \phi \longrightarrow \psi$. Then there is a formula $\theta \in L_\phi \cap L_\psi$ such that $\vdash \phi \longrightarrow \theta$ and $\vdash \theta \longrightarrow \psi$*

$\mathbb{SH}$: *Assume that $\vdash t^p(\phi) \longrightarrow \psi$. Then there is a natural number q and a formula $\theta \in L_\phi \cap L_\psi$ such that $\vdash t^q(\phi) \longrightarrow \theta$ and $\vdash \theta \longrightarrow \psi$*

$\mathbb{IS}4$, $\mathbb{IS}4$: *Assume that $\vdash \phi \longrightarrow \psi$. Then there is a formula $\theta \in L_\phi \cap L_\psi$ such that $\vdash \Box\phi \longrightarrow \theta$ and $\vdash \theta \longrightarrow \psi$*

$\mathbb{L}_n$: *Assume that $\vdash \phi \longrightarrow \psi$. Then there is a formula $\theta \in L_\phi \cap L_\psi$ such that $\vdash s_1\phi \longrightarrow \theta$ and $\vdash \theta \longrightarrow \psi$*

Proof. Apply interpolation to the push-out

$$\begin{array}{ccc} \mathcal{F}(Y) & \longrightarrow & \mathcal{F}(X\bigcup Y) \\ \uparrow & & \uparrow \\ \mathcal{F}(X\bigcap Y) & \longrightarrow & \mathcal{F}(X) \end{array}$$

where X and Y are sets, and $\mathcal{F}(X)$ is the free algebra (in the corresponding category) on X generators (thought of as propositional variables).

Remark 3.0.18 (i) Ordinary interpolation fails in $\mathbb{L}_3$. This is a consequence of the following counter-example to Beth's definability theorem: let $\sigma(p, p_1, p_2)$ be the formula $(s_1p \leftrightarrow p_1) \wedge (s_2p \leftrightarrow p_2)$. By Moisil's determination principle, $\vdash \sigma(p, p_1, p_2) \wedge \sigma(q, p_1, p_2) \longrightarrow (p \leftrightarrow q)$. If Beth's theorem were valid, p would be definable from p_1, p_2, i.e., there would be a formula $\theta(p_1, p_2)$ such that $\vdash \sigma(p, p_1, p_2) \longrightarrow (p \leftrightarrow \theta(p_1, p_2))$. If we replace (p_1, p_2) by (s_1p, s_2p), we obtain $\vdash p \leftrightarrow \theta(s_1p, s_2p)$. This is a contradiction, since s_ip is complemented and, by induction, so is $\theta(s_1p, s_2p)$. Thus, every formula would be complemented. The question whether ordinary interpolation holds for extensions of intuitionistic and modal S4 calculi has been extensively investigated. (See e.g. [10], [11] and [12] and the literature quoted in those works).

(ii) The previous question seems related to the following one: for which Heyting algebras with operators is (BC) for filters (of the appropiate kind) equivalent to (BC) for filters (of the same kind) generated by one element? Since interpolation was deduced precisely from (BC) for these filters (i.e., generated by one element), it would be interesting to give conditions on the algebras to assure the reverse implication, i.e., Interpolation implies (SA). We plan to return to this question in a forthcoming paper.

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