A mathematical analysis of Masaccio's *Trinity*

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The florentine Tommaso Cassai (1401-c.1427) [1] better known as Masaccio, has been hailed as the first great painter of the Italian Renaissance and his fresco *Trinity* (c.1425, Santa Maria Novella, Florence)[2] as the first work of Western art that used full perspective. It appears that the author was inspired and actually helped by his friend Filippo Brunelleschi (1377-1446), the celebrated architect of the cupola of Il Duomo di Firenzi (the cathedral of S. Maria del Fiore in Florence). It is less well-known that Brunelleschi was a pioneer in perspective and that he devised a method for representing objects in depth on a flat surface by using a single vanishing point.

The aim of this note is to study several questions of a mathematical nature suggested by this fresco:

(1) How accurate is the perspective of the fresco?

(2) What are the dimensions of the chapel?

(3) What are the dimensions of the coffers of the vaulted ceiling of the chapel?

(4) Where is the point of view situated with respect to the fresco?

(5) Where are the different characters situated inside the chapel?

(6) What are the "real" heights of the characters portrayed?

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[1] Actually born near Florence, in San Giovanni Valdarno

[2] See http://www.kfki.hu/ arthp/html/m/masaccio/trinity/trinity.html

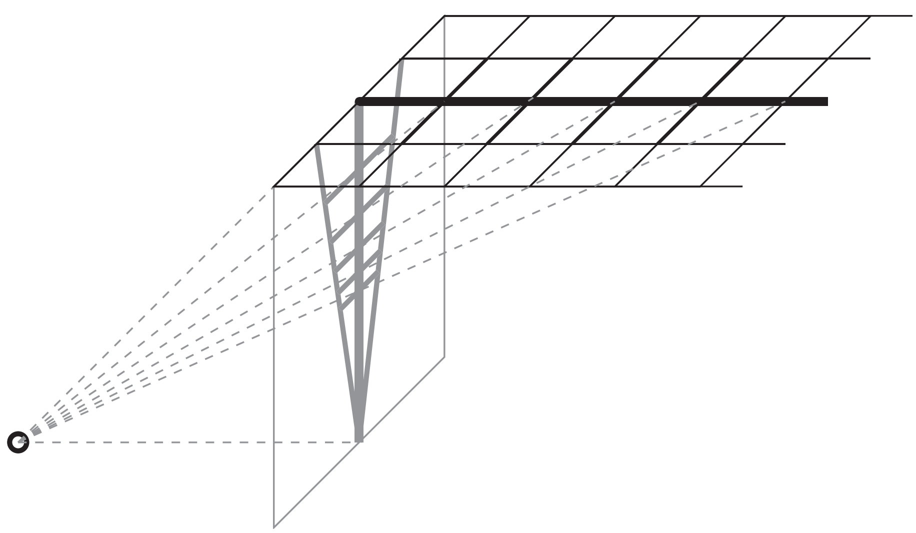
Questions (1)-(4) admit answers that may be computed starting from the data of the fresco, by using some rules of perspective and simple mathematical facts. This is not true for the others. Nevertheless, we will show that under some reasonable hypotheses estimates may be made.

The mathematical methods used are elementary and were known to Euclid. So they were accessible to Masaccio. To make the text more readable, mathematical developments are relegated to the Appendix. All the figures are given in cm.

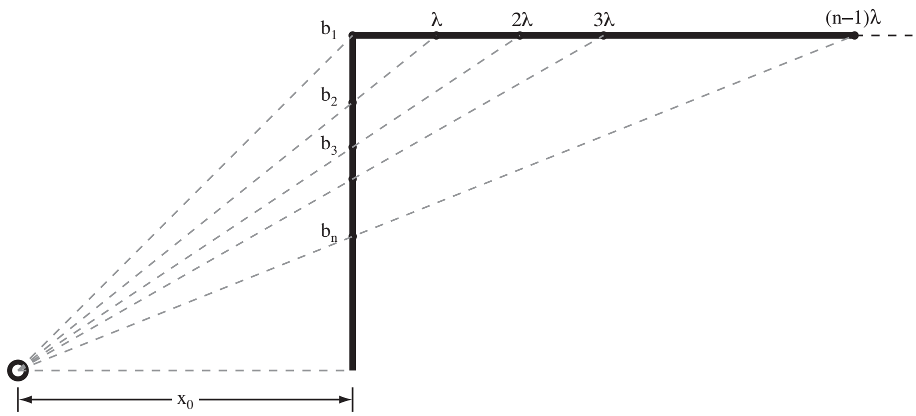
1 Checking the data

We will check whether the data provided by [1] (page 100) fits the theoretical criterion developed in the Appendix (Section 5.1 (3).) We let b_n ($n < 10$) be the distance from the line of the reclining figures to the intersection of the n^{th} circle (starting from the top) with the line of symmetry, i.e., the vertical line in the painting going through the vanishing point.

To understand what is going on, we look at the simpler problem of the representation of a plane, rather than cylindric vault.



From this diagram, we just keep "its dorsal spine" (i.e. the bold line), which is all what the data in [1] (page 100) is about and which is the same for both vaults (plane and cylindric)



The following data ("empirical b_n") is either given explicitly in the above reference or follows from additions and subtractions from the data therein. We will refer to these data as "robust" and use the term "fragile" to our own measures on magnified copies of Xerox reproductions of pictures in books or WEB.

The values obtained from the criterion ("theoretical b_n") are computed in the Appendix (Section 5.1.)

n	empirical b_n	theoretical b_n	*error*
1	416.80	416.80	0.00%
2	393.06	393.06	0.00%
3	371.95	371.88	0.02%
4	352.46	352.86	0.11%
5	335.67	335.70	0.01%
6	319.63	320.12	0.15%
7	305.95	305.93	0.01%
8	292.72	292.94	0.01%
9	280.92	281.01	0.03%

The fit is excellent. In all cases the error is less than two parts in a thousand. This shows that Masaccio constructed a projection from a point on the line situated on the horizontal plane of the reclining figures, perpendicular to the horizon and going through the point of intersection of the horizon and the line of symmetry in the fresco.

On the other hand the magnified copy [3] would lead us to think that the "rays" of the vault do not converge to this point, but rather to a point below.

This is certainly wrong, as shown by the fit of the data with the theoretical analysis. It is possible that the damage caused to the fresco by removing it from the original place, putting it back again, covering by an altar (by Vasari) and renovating it in a period when no sound scientific basis for this operation was known, may account for this and other mismatches that we shall point out later on.

In the Appendix (Section 5.1) it is shown that the distance from the point of view to the entrance of the chapel is

$$x_0 = 16.56\lambda$$

where λ is the length of an individual coffer. Similarly, the length of the interior of the chapel, i.e., the space under the coffers, may be expressed in terms of λ: since there are 7 rows of coffers, each of length λ,

$$\text{length of the interior of chapel} = 7\lambda$$

The other dimensions are given in the data [1]:

$$\text{height of the chapel} = 416.8^3$$

$$\text{width of the chapel} = 211.6$$

This solves the problem of the dimension of the chapel *provided that* λ *may be computed.*

2 Dimensions of the chapel

As we pointed out in the remarks, we need only to compute the length of an individual coffer. But first, we tackle the question of the width. The width of an individual coffer may be computed quite easily. In fact, since the radius of the "virtual" cylinder is given in the data of [1], namely 105.8, the total length of the frontal arch is $L = \pi \times 105.8 = 332.38$. Since there are 8 rows of coffers, the width of each is $w = 323.38/8 = 41.55$. On the other hand, the length is not so straightforward, The trouble is that although we can measure heights and widths, we cannot measure depths. However, there is

[3] measured from the horizon rather than from the floor of the chapel

one object whose depth may be computed on the base of the given data: the square abacus on top of the columns. This gives the missing clue to compute depths.

To compute the length λ of an individual coffer we use both front and back columns and make the following

Assumption 1 *The abacus on top of the four columns under the arches (the two in front and the two in the back) are squares. More precisely, the top of the capital of each of the four columns is a square.*

This supposition is natural, since the columns are cylindric. Now the idea is to take the square abacus on top of a column as a "patron" or unit of measure for the depth of the chapel. The (real) length of this abacus can be measured directly from the fresco or rather inferred, since part of the horizontal side of the abacus (the one that can be measured) is hidden. The trouble is that the apparent length of the patron decreases as we take it along the "diagonal" between the top of the front column to the top of the corresponding back column in the fresco. But we can take averages. In details: if we imagine identical abacus between the columns, their number n is

$$\text{(app. distance between columns/average app. length of abacus)} + (a-\lambda)/a.$$

(The term $(a-\lambda)/a$ is due to the fact that the fraction $1/2((a-\lambda)/a)$ of the first abacus and the same fraction of the last abacus are inside the chapel). Since the apparent lengths of these abacus are known, the distance between the columns is roughly $n \times$ real length of abacus. On the other hand, this distance is $7 \times \lambda$ and this allows us to compute λ. In what follows a_f is the apparent length of the front top abacus and a_b the apparent length of the back front abacus.

	app. distance	a_f	a_b	average	a	λ
Left	5.5	1.2	0.9	1.05	42.16	32.87
Right	5.3	1.3	1.0	1.15	41.39	29.02

Without taking averages, we have the following inequalities corresponding to the apparent lengths of the front abacus (a_f) and the back abacus (a_b) for left and right columns, respectively

$$\begin{cases} 29.42 < \lambda < 37.48 \\ 26.27 < \lambda < 32.60 \end{cases}$$

We notice that we have a "robust" lower bound, in contrast to our "fragile" ones for λ, namely

$$\lambda > 23.74$$

To explain where this value comes from, look at the Appendix (Section 5.3). Unfortunately, we don't have a "robust" upper bound for λ.

It seems likely that nothing more precise may come out of these measures and that the value of λ is between 26.27 and 37.48. Correspondingly, the length of the interior of the chapel is between 183.89 and 262.36 and the distance x_0 from the viewpoint to the entrance of the chapel is between 435.03 and 620.67.

For definiteness sake we take $\lambda = 31$ (roughly the average of the values given by the above table) as the length of an individual coffer. As a consequence, we obtain

$$\begin{cases} \text{distance from the viewpoint to the chapel} & = & 513.36 \\ \text{length of the interior of the chapel} & = & 217 \end{cases}$$

From a "practical" point of view, the exact value of λ does not matter too much. If λ were 33, for instance, the length of the interior of the chapel would be 231, rather than 217, a difference of 14 cm. Now, we turn to problems (5) and (6).

3 Position of characters on the ground

The problem of finding the positions of the characters of the fresco on the ground of the chapel can not be solved by studying the fresco only and the measures therein and some external clues as well as some tinkering is needed to proceed. The reason is that, grosso modo, figures of different heights and situated in different places may have the same projection.

We have an historical clue: the height of Christ. According to J.A. Aiken, "...four columns formerly in the *aula del Concilio* of the lateran in Rome were believed in Masaccio's time to establish the height of Christ at approximately 1.78 m."([1]). Thus, we make the following

Assumption 2 *Masaccio took the real height of Christ to be 178 cm*

This assumption allows us to find the position of Christ (and the cross) inside the chapel. In fact, as shown in the Appendix (Section 5.2), the depth of Christ (i.e., the distance from the entrance of the chapel) is

$$d = 3.36\lambda.$$

For $\lambda = 31$, we obtain that Christ depth is 104.16. Thus, all the scene of the crucifixion with the Virgin and St. John as witnesses takes place in this rather reduced space. (Recall that the figures are *inside* the chapel and this leaves a space of 104.16-31=73.16 as the available space for the whole scene).

Had we taken the height of Christ to be three "braccia" (approximately 172), as the canon of what an ideal man should measure in the Renaissance, its depth would be 83.25, a figure that seems too small as a theater of the scene. We keep the first figure, the one given by the historical clue.

We next tackle the Father's depth. Although the apparent height may be measured directly on the fresco (155), some tinkering seems necessary to proceed as the notion of real height does not make sense. We shall concentrate on the distance between the Father and Christ. Notice that the Father is holding the cross with his arms extended in such a way that his hands are approximately 95 apart and this suggests a distance between the two not far from 10 or 15 from the cross. In fact, this seems to be a comfortable position to hold the cross. At any rate, we present a table for his depth and real height with different choices of separation between Christ and the Father around 10 or 15. Furthermore, we tabulate the distance between his head and the vault and the length of the support of the Father. Notice, however, that the height of the chapel is 416.8-26.44=390.37, since the first figure is the height measured from the horizon, i.e. the level of the kneeling figures, rather than the floor of the chapel. The step to go from that level to the floor is 26.44 from the magnified copy. Details of these calculations are in Appendix (Section 5.2)

separation	d	d/λ	real height	head/vault	length support
0	104.16	3.36	186.46	52.38	174.84
5	109.16	3.52	187.97	49.65	169.84
10	114.16	3.68	189.48	46.91	164.84
15	119.16	3.84	190.99	44.18	159.84
20	124.16	4.01	192.50	41.44	154.84
25	129.16	4.17	194.01	38.71	149.84
30	134.16	4.33	195.52	35.98	144.84

For the separation 5, the Father would be directly in the middle of the chapel; for the separation 15, his height would be three and a third bracia (florentine measure=57.33cm). The majority of people I showed the picture chose a point in the vault between the third and the fourth coffers and rather closer to the fourth as lying directly above the Father. This choice corresponds to a separation between 5 and 20 and closer to 20.

Now we tackle the depths of the Virgin and St. John. First, notice that they seem to be approximately the same and both are standing just before Christ, their separation to the cross being not far from 15, say. The following is a a table for their depth and real heights around this figure

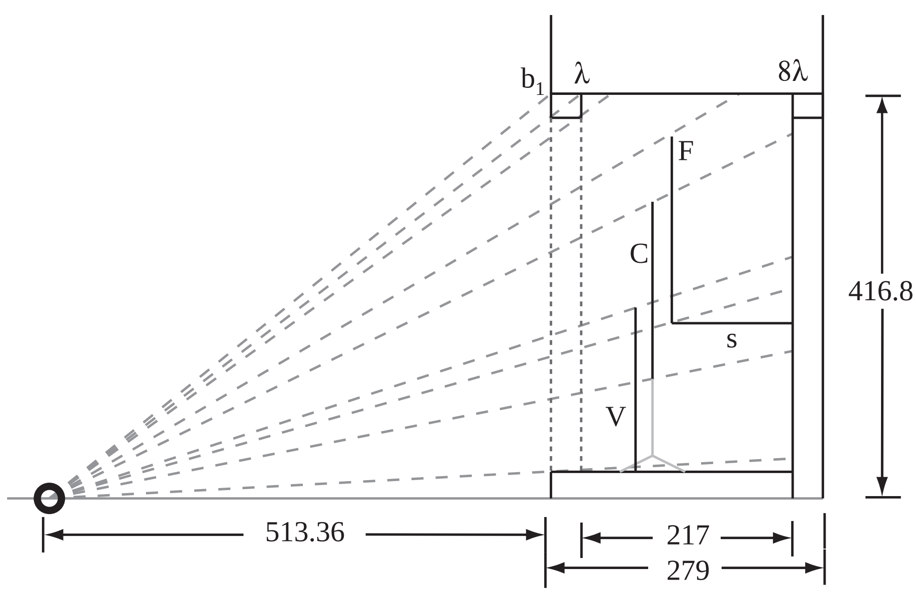
separation	depth	real height Virgin	real height St. John
0	104.16	166.40	162.74
5	99.16	165.06	161.42
10	94.16	163.71	160.10
15	89.16	162.37	158.78
20	84.16	161.02	157.47
25	79.16	159.68	156.15
30	74.16	158.34	154.83

Unfortunately, we don't seem to have records of their believed heights, contrary to the case of Christ. On the other hand, the average height of a man in the period was about 160.

4 Conclusions

The questions raised could be divided roughly into three types: those that can be answered from measures performed on the fresco itself; those that can be answered using historical clues and finally, those whose answers can only be guessed from common sense. The question of the accuracy of the perspective in so far as the apparent decreasing distances between the vault ribs may be answered quite precisely and we have done so. The error between theory and practice is less than two parts in one thousand. The question of the dimensions of the chapel may be answered in principle from measures performed on the fresco. The only problem is the accuracy of the measures. We should notice, however, that there seem to be discrepancies between measures that in theory should be the same. Such is the case with the heights of the front columns and the size of the abacus on top of them. As we said before, it is likely that the fresco has been badly damaged after the different transformations that it suffered: changes of place in the church and renovations in a period when there was no sound scientific base for this operation. (For the saga of the fresco, see [2]). Finally, the question of the position of the characters inside the chapel require some external clues, such as the historical clue on the height of Christ and some tinkering about the relation between Christ and the rest of the figures.

We sum up our conclusions in the form of a diagram with 20 as the distance between Christ (C) and the Father (F) who stands on the support (s), and as the difference between the depths of Christ and the Virgin (V).



One mystery remains: what is the length of the Father's support? From the assumption that the distance between Christ and the Father is about 20, it follows that the length of the support *inside the chapel* is about 124 and the whole support is 154.84, a figure that seems excessive. Most people I have asked the question gave answers of the order of 80 for the whole support and, consequently, about 40 for the part of the support inside the chapel, although one person, a painter, suggested 150 for the length of the whole support. Given the precision of the painting, we are inclined to think that Masaccio did this on purpose, as if he would like to leave the mystery of the Trinity untouched at this visual level.

5 Mathematical Appendix

5.1 A theoretical criterion

To solve the first question, we first formulate a purely mathematical criterion for the existence of a projection centered on the line situated on the horizontal plane of the reclining figures, perpendicular to the horizon and going through the point of intersection of the horizon and the line of symmetry in the fresco

Proposition 1 *Let $b_1, b_2, b_3, \ldots, b_n, \ldots$ be a strictly decreasing sequence of positive real numbers. Then the following are equivalent:*

(1) For every $\lambda > 0$, there is a unique point $(x_0, 0)$ on the x-axis such that the projection of the y-axis on the line $y = b_1$ from this point projects $(0, b_n)$ into $(-(n-1)\lambda, b_1)$. In symbols: $(0, b_n)\overline{\wedge}(-(n-1)\lambda, b_1)$.

(2) For every natural number $n \geq 1$

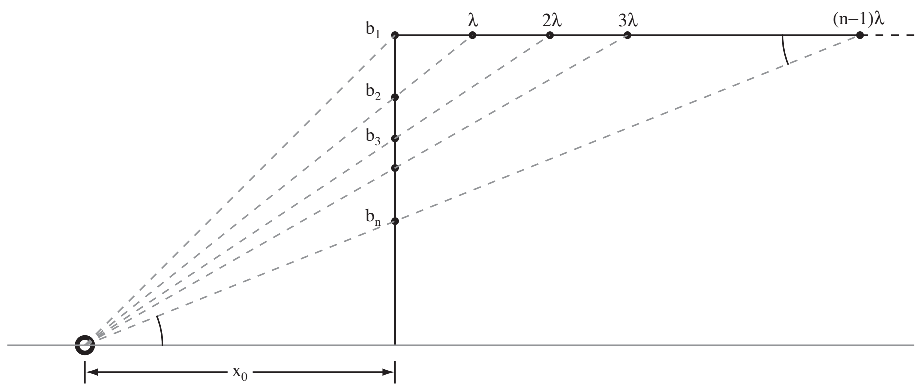
$$b_n = b_1/[1 + (n-1)\omega]$$

where $\omega = (b_1 - b_2)/b_2$

Proof:

(1)⇒(2) : By similarity of triangles in the diagram below

$$\begin{cases} b_n/x_0 &= (b_1 - b_n)/[(n-1)\lambda] \\ b_2/x_0 &= (b_1 - b_2)/\lambda \end{cases}$$



Dividing the first equation by the second and isolating b_n we obtain the desired formula.

(2)⇒(1) : Let λ be an arbitrary positive real number. We define

$$x_0 = \lambda/\omega.$$

We have to show that

$$(0, b_n)\overline{\wedge}(-(n-1)\alpha, b_1)$$

In other words, we have to show that the intersection of the line l_n joining $(x_0, 0)$ and $(0, b_n)$ with the line $y = b_1$ is the point $(-(n-1)\alpha, b_1)$. First notice that the equation of l_n is $y = -(b_n/x_0)x + b_n$. A simple computation shows that the lines in question meet at the point $((b_n - b_1)x_0/b_n, b_1)$. But it follows from (2) that $(b_1 - b_n)/b_n = (n-1)\omega$. Replacing x_0 by λ/ω, we obtain the desired result. Uniqueness of x_0 is obvious.

Notice that (2) implies that all the b_n's are known once that we know the first two of them.

Remark Although we don't need them, we may add without proof the following equivalent conditions to (1) and (2):

(3) If n_1, n_2, n_3, n_4 are natural numbers such that $n_1 \neq n_4$ and $n_2 \neq n_3$, the cross-ratio of the corresponding $b's$ is given by

$$(b_{n_1} b_{n_2} b_{n_3} b_{n_4}) = (n_1 - n_3)/(n_1 - n_4) \times (n_2 - n_4)/(n_2 - n_3)$$

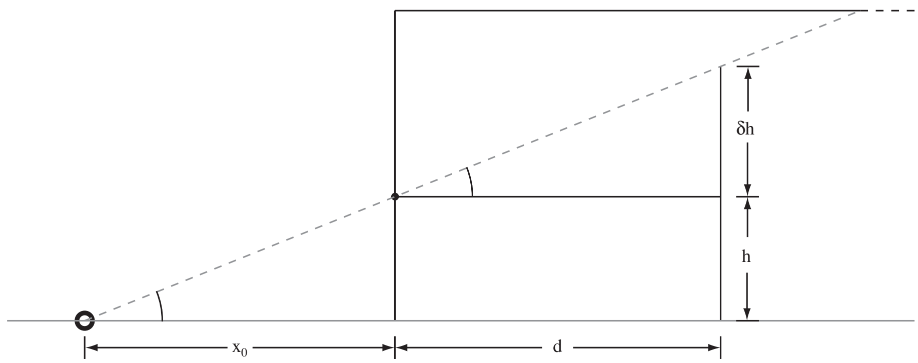
Furthermore, $lim_{n\to\infty} b_n = 0$

(4) For every m, n

$$(b_n - b_m)/b_n b_m = (m - n)(b_1 - b_2)/b_1 b_2$$

5.2 Apparent vs. real lengths

Let us call the depth of A the distance from the front of the chapel to A. From the figure



we deduce (from similarity of triangles) the formula for the depth of a vertical segment A

$$d = x_0 \delta h / h = \frac{16.56 \lambda \delta h}{h}$$

where h the apparent height (measured on the painting) of A and δh the difference between the real height an the apparent height of A.

Since the apparent height of Christ is $h = 148$, $\delta h = 30$ and it follows from this formula that the depth of Christ is 3.36λ. For $\lambda = 25.68$, we obtain that Christ depth is 86.28.

5.3 A robust lower bound

To fully explain this value and provide further information that may be useful, we assume that the arches of the vault of the fresco are represented by the nine circles $C_1, C_2, C_3, \ldots, C_9$ shown in [1], page 100. The fact that the real arches that we assume are circular are represented by circles follows from the fact that the visual circular cone of the painter, whose directrix is the frontal arch, is cut by the vertical fresco in a circle, an elementary fact known to Apolonius. To compute the radius of these circles, let $(0, k_n)$ be the center of the circle C_n. By similarity of triangles, $r_n/k_n = r_1/k_1$. Fortunately, $r_1 = 105.8$ is given as data and k_1 is easily calculated from the data (by additions and subtractions) in [1] (page 100). In fact, $k_1 = 311.2$. Thus, $r_n/k_n = 0.34$. We may also notice that $b_n - k_n = r_n$. Thus, $r_n = b_n - r_n/0.34$ and this formula may be rewritten as

$$r_n = 0.34 b_n/1.34$$

Using this formula, $r_1 = 0.34 \times 416.8/1.34 = 105.76$ which is very near the value 105.8 given in the data (error: 0.04%).

The apparent width of a coffer on the n^{th} rib is $w_n = r_n \times \pi/8$ and its apparent length is $\lambda_n = b_n - b_{n+1}$. The ratio between the apparent width and the apparent length of a coffer on the same rib is $\phi_n = w_n/l_n$. We organize these values in the following table

n	r_n	λ_n	w_n	ϕ_n
1	105.76	23.74	41.55	1.75
2	99.73	21.11	39.16	1.86
3	94.38	19.49	37.06	1.90
4	89.43	16.79	35.12	2.09
5	85.17	16.04	33.45	2.09
6	81.10	13.68	31.85	2.33
7	77.63	13.23	30.49	2.31
8	74.27	11.80	29.17	2.47
9	71.28	-	28.00	-

Clearly, $\lambda > \lambda_1 = 23.74$, the lower bound mentioned in the text. Furthermore, $\phi < \phi_1 = 1.75$.

6 A comparison with the results of the crs4 Group

After finishing this paper, my attention was called to a WEB site ([5]) where a reconstruction of Masaccio's chapel is attempted. Unfortunately, there are no details, no statement on their assumptions and the phrase "From the geometry it is actually possible to work backwards to reconstruct the full volume in measured accuracy of the 3-dimensional space Masaccio depicts" gives the erroneous impression that no further assumptions are needed for the reconstruction. The final result is given by means of a drawing (see http://www.crs4.it/Ars/arsgifs/zmasacciodiagram.gif). The measures below are based exclusively on this drawing and hence are very rough. I computed the scale using the measures in [1]. In the column "this paper" I took 20 as the distance between Christ and the Father and as the difference between the depths of Christ and the Virgin and approximate to .5.

	crs4 Group	this paper
λ	32	31
x_0	$544 = 17\lambda$	$513.5 = 16.56\lambda$
length chapel	$224 = 7\lambda$	$217 = 7\lambda$
height Christ	179	178
height Father	192	192.5
head/vault	32	41.5
height Virgin	-	164
height St. John	166.5	160.5
length support Father	147	155
depth Christ	$134.5 = 4.2\lambda$	$104 = 3.36\lambda$
depth Father	$160 = 5\lambda$	$124 = 4\lambda$

There is considerable overall agreement, although with some important differences. It is interesting to note the coincidence on the real height of Christ and I suspect that they took this figure, just as we did, as an assumption for their reconstruction. Furthermore, there is no difference about λ since only a range of values around 30 was determined by the fresco and I took 31 for definiteness sake.

The main discrepancy is about x_0, the distance from the viewpoint to the chapel. This distance can be *proved* to be 16.56λ. (as we did in the Mathematical Appendix), rather than 17λ as they suggest. This discrepancy accounts for the difference of depths of Christ and the other figures. At the depth they suggest, the height of Christ would be 185.51. A more precise comparison can only be made when the details of their work will be available.

Acknowledgments

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