Generic figures and their glueings

A constructive approach to functor categories

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Dedicated to the memory of John Macnamara
1929 – 1996

Contents

Introduction

Although there are several textbooks on topos theory, we feel that ours fills a definite need.

The fully fledged notion of a Grothendieck topos seems formidable to the beginner. Too many notions are invoked in its definition and most of the examples presuppose a good deal of mathematical experience. This makes the subject difficult both for beginner mathematicians as well as for people like logicians, linguists and psychologists who would like to know what topos theory is about and how to use it as a tool in their work.

The usual alternative is to start with the axiomatic approach to topos theory, delving into elementary toposes from the beginning. Although a lot of theory can be learned in this way, this approach has the disadvantage that one doesn't have the main examples of which elementary topos are an axiomatization of. In fact, we feel that logically as well as psychologically, well chosen examples are needed to understand an axiomatization. But elementary toposes are an axiomatization of Grothendieck toposes and we are back to the previous difficulty!

A few years ago, Lawvere suggested another alternative: to introduce topos theory through presheaf toposes or, equivalently, $\mathbb{C}$-sets. These are categories whose objects result from the glueing of simpler ones, the generic figures. These categories are Grothendieck toposes which do not involve the notion of a Grothendieck topology, making them much easier to understand and work with them.

This approach has several advantages, the most important being the simplicity of the categories involved and the rich theory to which they give rise. Several phenomena which distinguish toposes from the ordinary category of sets appear already at this level. In fact, some of these toposes have played an important role in the search of further axioms on a topos to define toposes of space, toposes of motion, etc.

Although Lawvere and Schanuel developed this approach in their beautiful book ([27]), the scope of that work did not allow them to go into a systematic study of presheaf toposes and their connections.

Our book follows this alternative to its bitter end. After the definition of a category of $\mathbb{C}$-sets we consider six easy to understand examples which,

furthermore, have clear graphical representations: categories of sets, bisets, bouquets, graphs, reflexive graphs and evolutive sets. They keep us company throughout the whole book to illuminate new material, interpret general results and suggest new theorems.

The description of our aims implicitly defines the reader that we have in mind: a beginner mathematician or scientist or philosopher (of an arbitrary age) who would like to take advantage of the rich structure and theory that presheaf toposes have to prepare himself or herself either for further study or for applications of the theory described.

We mentioned the book by Lawvere and Schanuel. This is an excellent work to learn the basic notions of category theory with well chosen examples and clear motivations. It is indeed a first introduction to categories. Our book presupposes the subject covered by theirs and prepares the reader to study more advanced works like the book by MacLane and Moerdijk (see [29]), a very readable account of topos theory that, however, can hardly be described as 'A first introduction' as the authors advertise it.

After describing the 'topos' of this book in the literature, a few words are in order to motivate $\mathbb{C}$-sets.

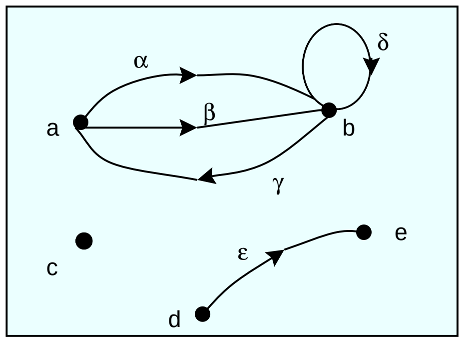
Assume that we want to describe and study a class of complicated structure, say graphs. Of course, a graph G is a collection of vertices and arrows with some relations of incidence between them. Usually, a graph is formalized as a couple of sets G_1 (the arrows) and G_0 (the vertices) together with two maps $s : G_1 \longrightarrow G_0$ (source) and $t : G_1 \longrightarrow G_0$ (target). These maps associate to an arrow its source and its target, respectively.

There is an alternatively, and we hope to show, more illuminating way of considering a graph, namely as consisting of figures of two shapes; the V-figures or figures of shape V ('vertex') and A-figures or figures of shape A ('arrow') subject to some relations that we describe in terms of 'change of figures'. Looking at the arrow itself as a graph, we see that it has exactly one A-figure (the arrow) and two V-figures (the source and the target). In a similar vein, we may look at the vertex as a graph with one vertex and no arrows. Thus, these figures that we called 'generic' constitute a category

$$V \underset{t}{\overset{s}{\rightrightarrows}} A$$

(with identities omitted). Each morphism is called a 'change of figure'. For instance $s : V \longrightarrow A$ allows to change an A-figure into a V-figure, namely into the V-figure which is the source of the original A-figure.

Incidence relations may be formulated in terms of right actions. To make this discussion more explicit, consider the following graph



We have 5 A-figures (or arrows): $\alpha, \beta, \gamma, \delta, \epsilon$ and 5 V-figures (or vertices): a, b, c, d, e with the following table of right actions:

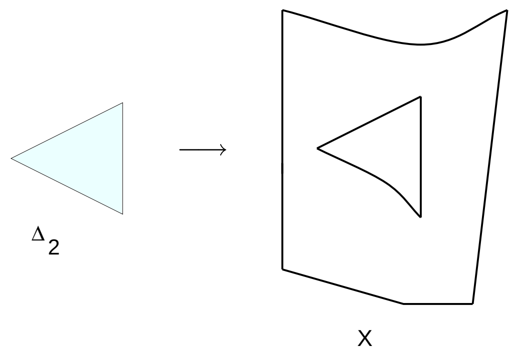
$$\begin{aligned}
&\alpha.s = \beta.s = \gamma.t = a \\
&\alpha.t = \beta.t = \gamma.s = \delta.s = \delta.t = b \\
&\epsilon.s = d \\
&\epsilon.t = e
\end{aligned}$$

The incidence relations can be read at once from the table. Notice, furthermore that in an obvious sense, the whole graph may be obtained by glueing its generic figures.

A morphism of graphs is a function that sends vertices into vertices, arrows into arrows and preserves the incidence relation, or what amounts to the same, the right action.

This way of analyzing complex mathematical structure in terms of simpler ones (generic figures) with a right action of change of figures has been implicit for a long time in mathematical practice, but it is only in our century that it was made explicit. In the so-called singular theory in Algebraic Topology, for instance, a topological space is viewed as consisting of figures 'points', 'intervals', 'triangles' and so on with changes of figures given by 'extracting the end points of an interval' or 'extracting the sides of a triangle' and so

on. In this case, the generic figures may be objectivized as actual continuous maps from the corresponding euclidean spaces into the space:



In a figurative way of talking, such a map is an 'extraction' of the euclidean figure from the space. The functors of singular homology and singular cohomology may be defined in this context.

To keep this intuition of 'extraction of generic figures', we shall use an alternative notation for figures and actions, two examples of which are given for the graph:

$$
\begin{tikzcd}
& A \arrow[r, dashed, "\alpha"] & X \\
V \arrow[ur, "s"] \arrow[urr, dashed, "\alpha.s=a"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[r, dashed, "\alpha"] & X \\
V \arrow[ur, "t"] \arrow[urr, dashed, "\alpha.t=b"'] & &
\end{tikzcd}
$$

Later on we shall see how, thanks to Yoneda lemma, these dotted arrows may be interpreted as real morphisms, making the analogy with singular theory very close.

A few words about the contents: the book is divided into 14 chapters. The first five deal with categories of $\mathbb{C}$-sets. We study operations on $\mathbb{C}$-sets and the object Ω of truth values. After two chapters on adjointness in posets and in categories, we return to $\mathbb{C}$-sets to deal with logical operations definable in them. This chapter is followed by one on doctrines, i.e., categorical counterparts of classical and non-classical logics. The next three chapters deal with geometric morphisms which are the objective way to describe connections between categories of $\mathbb{C}$-sets. The last chapter deals with points of categories of $\mathbb{C}$-sets and their connection with models of some infinitary theories described in terms of the category of figures.

There are exercises in every chapter. They are important and several are presupposed later in the main text. Some chapters have 'windows'. These are designed for the reader to have a glimpse on other territories not covered in the main text. Some are mainly bibliographical, others frankly philosophical.

This book originated in a two-year course (1991-1993) that Houman Zolfaghari taught (from the point of view of generic figures and their glueings) for the benefit of a multidisciplinary group working on logical foundations of cognition. Besides the authors of this book, the group included John Macnamara. We felt that all of us should have a working knowledge of the tools that we were using in our work, rather than counting on the usual division of labour. This was a challenge since Macnamara, a psychologist at McGill, had never studied mathematics before. The result was very encouraging. We will never forget his joy in realizing that he could compute geometric morphisms between the main examples. As he put it, he was promoted from a christian to a lion.

During Reyes sabbatical year (1996/97) from Montreal University, the first two authors had an opportunity to try the notes of Zolfaghari's course taken by the first author both in La Plata and in Santiago. The enthusiastic reception of the audience in La Plata launched us in the actual writing of this book.

It is a pleasure to thank people who have helped us during the long time of gestation.

First, our thanks go to Adriana Galli and Marta Sagastume for inviting Reyes to La Plata during the month of June 1996. Many thanks also to Renato Lewin who was responsible for the invitation of Reyes to the Pontificia Universidad Catolica de Santiago and for organizing a one year course on generic figures and their glueings at that university. The notes that the first author took in La Plata and in Santiago were the first version of this book. Among the participants to both courses we would like to mention, besides the organizers, Matias Menni, Hector Gramaglia, Gaston Argeri, Guillermo Ortiz and Carlos Martinez who, through many questions and remarks, forced us to better explain, change and improve the presentation in a number of places.

Last, but not least, we are in great debt to Bill Lawvere. We have freely used materials from his papers, books, talks and conversations. Furthermore his encouragement has meant a lot to us.

We dedicate this book to the memory of our friend John Macnamara.

The first author is grateful for the support from a grant of the Fonds du Québec pour la formation de chercheurs et l'aide à la recherche (FCAR) given to Macnamara and Reyes. The second author would like to thank the following institutions that made the actual writing possible: the Université de Montréal, for granting him a sabbatical leave, the Universidad de La Plata, for inviting him on two occasions as Visiting Professor, La Fundación Andes de Chile for helping him with financial support, the Pontificia Universidad Católica de Chile for appointing him Visitor Professor and offering him its facilities during his sabbatical leave and Canada's National Science and Engineering Research Council (NSERC) for maintaining its financial support during the time in which this book was written.

Montréal, spring 1998

Revised version: Montréal, spring 2004

1 The category of $\mathbb{C}$-*Sets*

Let $\mathbb{C}$ be a category whose objects are thought of as generic figures and whose morphisms are thought of as changes of figures. A $\mathbb{C}$-*Set* is a family $X = (X(F))_{F \in Ob(\mathbb{C})}$ of sets $X(F)$ indexed by the objects of $\mathbb{C}$, which we call *F-figures of X*, together with a right action:

$$(\sigma \in X(F), F' \xrightarrow{f} F \in \mathbb{C}) \mapsto \sigma.f \in X(F')$$

which satisfies

$$\sigma.1_F = \sigma$$

and

$$(\sigma.f).g = \sigma.(f \circ g)$$

Notation: We will use the following notations for an F-figure of X :
'$F \overset{\sigma}{\dashrightarrow} X$' which stresses the aspect 'extraction' of a figure of X or
'$\sigma \in_F X$' which makes clear the intuition that σ belongs to the container at the level F. We will also use the following diagram for the action of an arrow on a F-figure:

$$\begin{tikzcd} & F \arrow[r, dashed, "\sigma"] & X \\ F' \arrow[ur, "f"] \arrow[urr, dashed, "\sigma.f"'] & & \end{tikzcd}$$

A morphism $X \xrightarrow{\Phi} Y$ of $\mathbb{C}$-*Sets* is a rule which sends F-figures of X into F-figures of Y and which is compatible with the change of figures, i.e., such that $\Phi(\sigma.f) = \Phi(\sigma).f$. In other words 'a change of figure followed by a change of container ($\Phi(\sigma.f)$) is the same as a change of container followed by a change of figure ($\Phi(\sigma).f$)'. The following diagram will often be used in this context

$$\begin{tikzcd} & F \arrow[r, dashed, "\sigma"] & X \arrow[r, "\Phi"] & Y \\ F' \arrow[ur, "f"] \arrow[urr, dashed, "\sigma.f"'] & & & \end{tikzcd}$$

A more standard way of saying the same is that a $\mathbb{C}$-*Set* is a functor from the dual of $\mathbb{C}$

$$X : \mathbb{C}^{op} \longrightarrow Sets$$

and a morphism of $\mathbb{C}$-*Sets* is a natural transformation, i.e., Φ is a family of functions $\Phi = (\Phi_F)_{F \in Ob(\mathbb{C})}$ such that $\forall F' \xrightarrow{f} F \in \mathbb{C}$ the following diagram is commutative

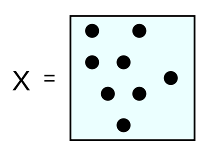
$$\begin{tikzcd}
X(F) \arrow[r, "\Phi_F"] \arrow[d, "(\,).f"'] & Y(F) \arrow[d, "(\,).f"] \\
X(F') \arrow[r, "\Phi_{F'}"'] & Y(F')
\end{tikzcd}$$

namely that $\Phi_F(\sigma).f = \Phi_{F'}(\sigma.f)$. We let $Sets^{\mathbb{C}^{op}}$ be the category of contravariant functors and natural transformations, i.e., the category of $\mathbb{C}$-sets. Such a functor is called a *presheaf* on $\mathbb{C}$ and, accordingly, $Sets^{\mathbb{C}^{op}}$ is called the *category of presheaves* on $\mathbb{C}$.

1.1 Examples of $\mathbb{C}$-*Sets*

- *Sets*. We can represent an object graphically and as a container .

Graphically:



A morphism between two sets is a function that sends points into points.

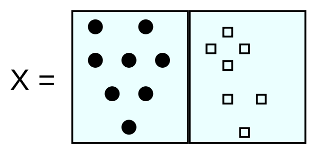
As a container: take $\mathbb{C} = \mathbb{1}$, the category containing only one generic figure P (for *point*), with the trivial change of figure 1_P. A set X may be identified with the $\mathbb{1}$-set or container whose P-figures (or points) are the elements of X : $\dfrac{P \overset{\sigma}{\dashrightarrow} X}{\sigma \in X}$ with the (trivial) action

$$\begin{tikzcd}
 & P \arrow[r, dashed, "\sigma"] & X \\
P \arrow[ur, "1_P"] \arrow[urr, dashed, "\sigma.1_P = \sigma"'] & &
\end{tikzcd}$$

and a function can be identified with a morphism of $\mathbb{1}$-sets, i.e., a rule that sends P-figures or points into points (which automatically respects the trivial action).

• *Bisets*. An object X is a couple of sets (X_0, X_1) and a morphism $X \xrightarrow{f} Y$ is a couple of functions (f_0, f_1) such that $X_0 \xrightarrow{f_0} Y_0$ and $X_1 \xrightarrow{f_1} Y_1$ are ordinary functions.

Graphically:



A morphism between two objects is a couple of functions that sends points into points and squares into squares.

As a container: take $\mathbb{C} = \mathbb{2}$ the category containing two generic figures: P (for *point*) and S(for *square*) and the identities 1_P and 1_S as the only morphisms. A biset X may be identified with the $\mathbb{2}$-set or container whose P-figures are the elements of X_0 and whose S-figures are the elements of X_1:

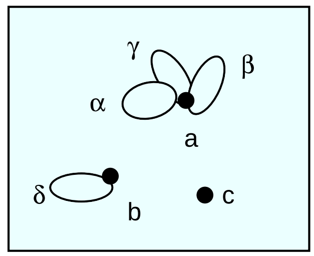
$$\frac{P \overset{\sigma}{\dashrightarrow} X}{\sigma \in X_0} \qquad \frac{S \overset{\sigma}{\dashrightarrow} X}{\sigma \in X_1}$$

with the trivial action. A morphism can be identified with a morphism of $\mathbb{2}$-sets, i.e., a rule that sends P-figures or points into points and S-figures or squares into squares (which automatically respects the trivial actions).

• *Bouquets*. An object X is a function $X_1 \xrightarrow{u} X_0$. We called the elements of X_1 *loops*, those of X_0 *vertices* and if α is a loop then $u(\alpha)$ is the *vertex* of the loop α. A morphism $X \xrightarrow{f} Y$ between $X = (X_1 \xrightarrow{u} X_0)$ and $Y = (Y_1 \xrightarrow{v} Y_0)$ is an ordered pair (f_1, f_0) where $X_1 \xrightarrow{f_1} Y_1$ and $X_0 \xrightarrow{f_0} Y_0$ are ordinary functions such that the folowing diagram is commutative

$$\begin{tikzcd}
X_1 \arrow[r, "u"] \arrow[d, "f_1"'] & X_0 \arrow[d, "f_0"] \\
Y_1 \arrow[r, "v"'] & Y_0
\end{tikzcd}$$

Graphically:

X = 

represents the bouquet $X_1 \xrightarrow{u} X_0$ where $X_1 = \{\alpha, \beta, \gamma, \delta\}$, $X_0 = \{a, b, c\}$ and $u(\alpha) = u(\beta) = u(\gamma) = a$, $u(\delta) = b$. For obvious reasons we call these equations *relations of incidence*. A morphism between two such objects is a function (or rule) that sends loops into loops, vertices into vertices and respects the actions or, as we will often say, preserves the incidence relations. Thus if $f = (f_1, f_0)$ is such a rule from $X_1 \xrightarrow{u} X_0$ into $Y_1 \xrightarrow{v} Y_0$, α is sent into a loop $f_1(\alpha)$ whose vertex is $f_0(a)$.

As a container: take $\mathbb{C} = V \xrightarrow{v} L$, the category having two generic figures: V (for *vertex*), L (for *loop*) and one morphism v (for *extraction of the vertex* of the loop). The identities 1_V, 1_L were omitted from the above representation of $\mathbb{C}$. The bouquet X may be identified with the $(V \xrightarrow{v} L)$-set whose V-figures are a, b, c and whose L-figures are α, β, γ δ:

$$\frac{V \dashrightarrow X}{a, b, c} \qquad \frac{L \dashrightarrow X}{\alpha, \beta, \gamma, \delta}$$

with obvious action. For instance

$$\begin{tikzcd} & L \arrow[r, dashed, "\alpha"] & X \\ V \arrow[ur, "v"] \arrow[urr, dashed, "\alpha.v=a"'] & & \end{tikzcd}$$

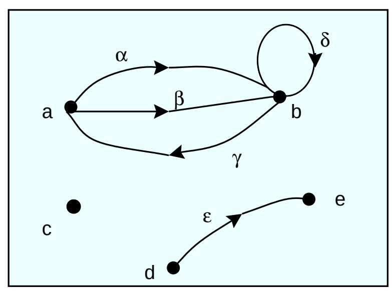
expresses the fact that the extraction of the vertex of α gives a. A morphism of $(V \xrightarrow{v} L)$-sets is a rule that sends V-figures into V-figures and L-figures into L-figures in such a way that the incidence relations are preserved.

• *Graphs*. Objects are oriented multi-graphs and morphisms are graph homomorphisms. In other words an object X consists of two sets X_1, X_0 and

two functions $X_1 \underset{u_1}{\overset{u_0}{\rightrightarrows}} X_0$. The elements of X_1 are called *arrows*, those of X_0 *vertices*, $u_0(\alpha)$ *the source* of α, $u_1(\alpha)$ *the target* of α. A morphism $X \xrightarrow{f} Y$ is a couple of functions $X_1 \xrightarrow{f_1} Y_1$, $X_0 \xrightarrow{f_0} Y_0$ such that the following diagrams commute, namely, $f_0u_0 = v_0f_1$, $f_0u_1 = v_1f_1$:

$$\begin{tikzcd} X_1 \arrow[r, shift left, "u_0"] \arrow[r, shift right, "u_1"'] \arrow[d, "f_1"'] & X_0 \arrow[d, "f_0"] \\ Y_1 \arrow[r, shift left, "v_0"] \arrow[r, shift right, "v_1"'] & Y_0 \end{tikzcd}$$

Graphically:

X = 

represents the graph $X_1 \underset{u_1}{\overset{u_0}{\rightrightarrows}} X_0$ where $X_1 = \{\alpha, \beta, \gamma, \delta, \epsilon\}$, $X_0 = \{a, b, c, d, e\}$ and $u_0(\alpha) = u_0(\beta) = a$, $u_0(\delta) = u_0(\gamma) = b$, etc., $u_1(\alpha) = u_1(\beta) = u_1(\delta) = b$, etc. A morphism between two such objects is a function (or rule) that sends arrows into arrows and vertices into vertices and preserves the incidence relations. Thus if $f = (f_1, f_0)$ is such a rule from $X_1 \underset{u_1}{\overset{u_0}{\rightrightarrows}} X_0$ into $Y_1 \underset{u_1}{\overset{u_0}{\rightrightarrows}} Y_0$, α is sent into an arrow $f_1(\alpha)$ whose source is $v_0(f_1(\alpha)) = f_0(a)$ and whose target is $v_1(f_1(\alpha)) = f_0(b)$.

As a container: take $\mathbb{C} = V \underset{t}{\overset{s}{\rightrightarrows}} A$, the category having two generic figures: V (for *vertex*), A (for *arrow*) and two morphisms s (for *extraction of the source* of the arrow) and t (for *extraction of the target* of the arrow). The identities 1_V, 1_A were omitted from the description of $\mathbb{C}$. The graph X may

be identified with the ($V \overset{s}{\underset{t}{\rightrightarrows}} A$)-set whose V-figures are a, b, c, d, e and whose A-figures are α, β, γ, δ, ϵ:

$$\frac{V \dashrightarrow X}{a, b, c, d, e} \qquad \frac{A \dashrightarrow X}{\alpha, \beta, \gamma, \delta, \epsilon}$$

with obvious action. For instance

$$\begin{tikzcd} & A \arrow[r, dashed, "\alpha"] & X \\ V \arrow[ur, "s"] \arrow[urr, dashed, "\alpha.s=a"'] & & \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[r, dashed, "\alpha"] & X \\ V \arrow[ur, "t"] \arrow[urr, dashed, "\alpha.t=b"'] & & \end{tikzcd}$$

expresses the fact that the extraction of the source of α gives a and its target is b. A morphism of ($V \overset{s}{\underset{t}{\rightrightarrows}} A$)-sets is a rule that sends V-figures into V-figures and A-figures into A-figures in such a way that the incidence relations are preserved.

• *Rgraphs*. Objects are oriented reflexive multi-graphs and morphisms are reflexive graph homomorphisms. In other words an object X consists of two sets X_1 , X_0 and three functions

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "u_0"] \arrow[r, "u_1"] & X_0 \arrow[l, shift left=2, "i"] \end{tikzcd}$$

with $u_0 i = u_1 i = 1_{X_0}$. The elements of X_1 are called *arrows*, those of X_0 *vertices*, $u_0(\alpha)$ *source* of α, $u_1(\alpha)$ *target* of α and $i(a)$ *distinguished loop* whose source (and target) is a.

A morphism $X \xrightarrow{f} Y$ where Y is

$$\begin{tikzcd} Y_1 \arrow[r, shift left=2, "v_0"] \arrow[r, "v_1"] & Y_0 \arrow[l, shift left=2, "j"] \end{tikzcd}$$

is a couple of functions $X_1 \xrightarrow{f_1} Y_1$, $X_0 \xrightarrow{f_0} Y_0$ such that the following dia-

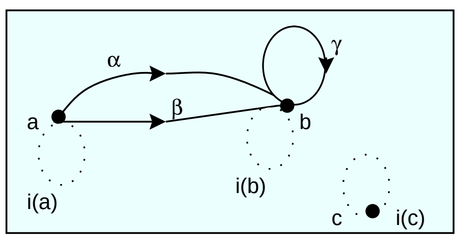
grams commute, namely, $f_0 u_0 = v_0 f_1$, $f_0 u_1 = v_1 f_1$ and $f_1 i = j f_0$.

$$\begin{tikzcd} X_1 \arrow[r, shift left, "u_0"] \arrow[r, shift right, "u_1"'] \arrow[d, "f_1"'] & X_0 \arrow[d, "f_0"] \\ Y_1 \arrow[r, shift left, "v_0"] \arrow[r, shift right, "v_1"'] & Y_0 \end{tikzcd}$$

and

$$\begin{tikzcd} X_1 \arrow[d, "f_1"'] & X_0 \arrow[l, "i"] \arrow[d, "f_0"] \\ Y_1 & Y_0 \arrow[l, "j"] \end{tikzcd}$$

Graphically:

X = 

represents the reflexive graph $\begin{tikzcd} X_1 \arrow[r, shift left=2, "u_0"] \arrow[r, "u_1"] & X_0 \arrow[l, shift left=2, "i"] \end{tikzcd}$ where

$$X_1 = \{\alpha, \beta, \gamma, i(a), i(b), i(c)\}, \quad X_0 = \{a, b, c\}$$

and $u_0(\alpha) = u_0(\beta) = a$, $u_0(\gamma) = b$, $u_1(\alpha) = u_1(\beta) = u_1(\gamma) = b$, $u_0 i(a) = u_1 i(a) = a$, etc. A morphism between two such objects is a function (or rule) that sends arrows into arrows, vertices into vertices, distinguished loops into distinguished loops and preserves the incidence relations. Thus if $f = (f_1, f_0)$ is such a rule from $\begin{tikzcd} X_1 \arrow[r, shift left=2, "u_0"] \arrow[r, "u_1"] & X_0 \arrow[l, shift left=2, "i"] \end{tikzcd}$ into $\begin{tikzcd} Y_1 \arrow[r, shift left=2, "v_0"] \arrow[r, "v_1"] & Y_0 \arrow[l, shift left=2, "j"] \end{tikzcd}$ $i(a)$ is sent into a distinguished loop $f_1(i(a))$ whose source (and target) is $f_0(a)$.

As a container: take $\mathbb{C} = \begin{tikzcd} V \arrow[r, shift left=2, "s"] \arrow[r, "t"] & A \arrow[l, shift left=2, "l"] \arrow[loop above, "\sigma"] \arrow[loop below, "\tau"] \end{tikzcd}$ the category having two generic figures: V (for *vertex*), A (for *arrow*) and morphisms s (for *extraction of the*

source of the arrow), t (for *extraction of the target* of the arrow) and l (for *extraction of the distinguished loop* from its vertex) such that $l \circ s = l \circ t = 1_V$, $\sigma = s \circ l, \tau = t \circ l$. Notice that we left identities out of the picture. The reflexive graph X may be identified with the ($\begin{tikzcd} V \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & A \arrow[l, shift left=3, "l"] \arrow[loop above, "\sigma"] \arrow[loop below, "\tau"] \end{tikzcd}$)-set whose V-figures are a, b, c and whose A-figures are α, β, γ, $i(a)$, $i(b)$ and $i(c)$:

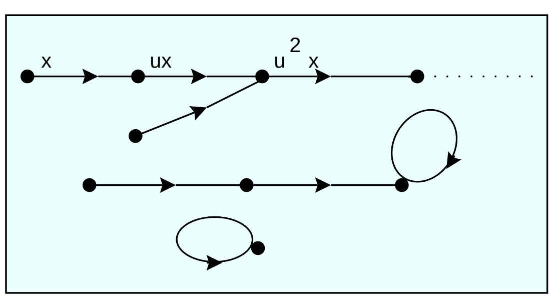
$$\frac{V \dashrightarrow X}{a, b, c} \qquad \frac{A \dashrightarrow X}{\alpha, \beta, \gamma, i(a), i(b), i(c)}$$

with obvious action. For instance the three diagrams

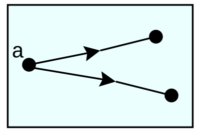
$$\begin{tikzcd} & A \arrow[r, dashed, "i(a)"] & X \\ V \arrow[ur, "s"] \arrow[urr, dashed, "i(a).s=a"'] & & \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[r, dashed, "i(a)"] & X \\ V \arrow[ur, "t"] \arrow[urr, dashed, "i(a).t=a"'] & & \end{tikzcd} \qquad \begin{tikzcd} & V \arrow[r, dashed, "c"] & X \\ A \arrow[ur, "l"] \arrow[urr, dashed, "c.l=i(c)"'] & & \end{tikzcd}$$

express three things: the extraction of the source of $i(a)$ gives a, the extraction of the target of $i(a)$ gives a and the extraction of the distinguished loop of c gives $i(c)$. A morphism of ($\begin{tikzcd} V \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & A \arrow[l, shift left=3, "l"] \arrow[loop above, "\sigma"] \arrow[loop below, "\tau"] \end{tikzcd}$)-sets is a rule that sends V-figures into V-figures and A-figures into A-figures in such a way that the incidence relations are preserved.

• *Esets*. Objects are evolutive sets or deterministic, discrete dynamical systems and morphisms are functions that respect the evolution. An *evolutive set* X is a pair (X_0, u) where X_0 is a set and $X_0 \xrightarrow{u} X_0$ an ordinary endofunction. A morphism $X \xrightarrow{f} Y$, where $Y = (Y_0, v)$ is a function $X_0 \xrightarrow{f} Y_0$ such that $f(u(x)) = v(f(x))$ for all $x \in X$. We call the elements of X_0 the *elements (or states)* of X and u the *evolution* of X. Graphically, an evolutive set is represented as follows:

X =


Note that

X = 

does not represent an evolutive set since the evolution of a should be uniquely determined.

As a container: we take as category of generic figures and changes of figures the category $\mathbb{E}$ having one object $*$ and the iterations of a morphism

$$\begin{tikzcd} * \arrow[loop above, "\sigma"] \end{tikzcd}$$

Thus, the morphisms are 1_*, σ, σ^2, σ^3,... In other words, $\mathbb{E}$ is the free monoid on one generator (σ) seen as a category. An evolutive set $X = (X_0, u)$ may be identified with an $\mathbb{E}$-set whose elements are the elements of X_0 and whose action is $x.\sigma^n = u^n(x)$:

$$\frac{* \overset{x}{\dashrightarrow} X}{x \in X_0} \qquad \begin{tikzcd} & * \arrow[r, dashed, "x"] & X \\ * \arrow[ur, "\sigma^n"] \arrow[urr, dashed, "x.\sigma^n = u^n(x)"'] & & \end{tikzcd}$$

(The equation for the action $x.\sigma^n = u^n(x)$ is forced by definition of an action and the particular case $x.\sigma = u(x)$.) A morphism of $\mathbb{E}$-sets $X \xrightarrow{f} Y$ is a function $X_0 \xrightarrow{f} Y_0$ which preserves the evolution in the sense that $f(x.\sigma^n) = f(x).\sigma^n$, i.e., $f(u^n(x)) = v^n(f(x))$ where $X = (X_0, u)$ and $Y = (Y_0, v)$. Notice that f preserves the action if and only if $f(u(x)) = v(f(x))$.

1.2 Monomorphism, epimorphism, isomorphism

The notions of the title of this section are well-known and may be defined in any category. For $\mathbb{C}$-sets they may be described as follows:

Proposition 1.2.1 *Let $f : Y \longrightarrow X$ be a morphism of $\mathbb{C}$-sets. Then*

(1) f is a monomorphism iff $f_C : Y(C) \longrightarrow X(C)$ is a (set-theoretical) injection, for every $C \in \mathbb{C}$

(2) f is an epimorphism iff $f_C : Y(C) \longrightarrow X(C)$ is a (set-theoretical) surjection, for every $C \in \mathbb{C}$

Remark 1.2.2 There is not a single, clear cut analogue of the notion of set-theoretical surjection in arbitrary categories. The 'weakest' analogue seems to be the notion of epimorphism. But other notions have been studied in the literature. Let us mention two: regular epimorphism and extremal epimorphism or surjections. Fortunately, all of these notions coincide in a category of presheaves (and, more generally, in a topos).

(3) f is an isomorphism iff $f_C : Y(C) \longrightarrow X(C)$ is a (set-theoretical) bijection, for every $C \in \mathbb{C}$

Proof.
We shall prove *(1)*, leaving the other as exercises.

$\rightarrow$: Assume that $Y \xrightarrow{f} X$ is a monomorphism. Let $C \in \mathbb{C}$ and $a, b \in Y(C)$ such that $f_C(a) = f_C(b)$. By Yoneda, we may identify a, b with morphisms of $\mathbb{C}$-sets with domain C and codomain Y such that $f \circ a = f \circ b$. By definition of monomorphism, $a = b$.

$\leftarrow$: Let

$$Z \overset{u}{\underset{v}{\rightrightarrows}} Y \xrightarrow{f} X$$

be a diagram such that $f \circ u = f \circ v$. Then, for each $C \in \mathbb{C}$, $f_C u_C = f_C v_C$. Let $z \in Z$. Then $f_C(u_C(z)) = f_C(v_C(z))$. Since f_C is an injection, $u_C(z) = v_C(z)$. But z was arbitrary and so $u_C = v_C$ for each component, i.e., $u = v$. $\Box$

In the category of sets there are canonical monomorphisms, namely inclusions, that 'represent' monomorphisms in the sense that for each monomorphism $f : Y \longrightarrow X$ there is a unique subset $A \overset{i}{\hookrightarrow} X$ and a unique bijection $Y \xrightarrow{g} A$ such that $f = ig$.

Following the analogy with sets, we first define the notion of 'sub-presheaf' or 'sub-$\mathbb{C}$-set', which generalizes the notion of subset.

A *sub-$\mathbb{C}$-set* of a $\mathbb{C}$-set X is a $\mathbb{C}$-set Y with a morphism of $\mathbb{C}$-sets $i : Y \hookrightarrow X$ such that for each $F \in Ob(\mathbb{C})$, $i_F : Y(F) \subseteq X(F)$ is the set inclusion. We sometimes will find more convenient to use the following equivalent definition: a *sub-$\mathbb{C}$-set* Y of a $\mathbb{C}$-set X is a family $Y = (Y(F))_{F \in Ob(\mathbb{C})}$ closed under the action, namely, if σ is an F-figure of Y (and hence of X) and $F' \xrightarrow{f} F$ is a morphism of $\mathbb{C}$, then the result of the action in the sense of X, $\sigma ._{X} f$ is an F'-figure of Y (and not only of X):

$$\begin{tikzcd} & F \arrow[r, dashed, "\sigma"] & Y \\ F' \arrow[ur, "f"] \arrow[urr, dashed, "\sigma._X f"'] & & \end{tikzcd}$$

As an example, a subgraph of a graph is a collection of arrows and vertices such that whenever an arrow belongs to the collection, its source and target also belong to the collection.

`EXERCISE 1.2.1`

(1) If $A \overset{i}{\hookrightarrow} X$ is a sub-$\mathbb{C}$ set of X, then i is a monomorphism. Conversely, if $f : Y \longrightarrow X$ is a monomorphism show the existence of a unique sub-$\mathbb{C}$ set A of X and a unique isomorphism $Y \xrightarrow{g} A$ such that $f = ig$.

(2) Let $\mathbb{C}$ be a small category. Show that in the category of $\mathbb{C}$-sets the isomorphisms are precisely the monomorphisms which are also epimorphisms.

(3) Let Δ_1 be the monoid of the endomorphisms of the arrow A of reflexive graphs. As a category, Δ_1 has one object and three arrows: 1, δ_0 and δ_1 with the relations $\delta_i \delta_j = \delta_i$ (where δ_0 is the constant map that sends the whole arrow in its source and δ_1 the arrow in its target.) Show that $Sets^{{\Delta_1}^{op}}$ may be identified with the category of reflexive graphs.

`EXERCISE 1.2.2`

If $\mathbb{C}$ is any of the categories below, interpret the category of $\mathbb{C}$-Sets both graphically and as a container:

(1)
$$\begin{tikzcd} A \arrow[r, "f"] & B \\ C \arrow[ur, "g"'] & \end{tikzcd}$$

(2)
$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "g"'] & A \\ & C \end{tikzcd}$$

(3) $$A \xrightarrow{f} B \xrightarrow{g} C$$

(4) $$\begin{tikzcd} V \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A \arrow[r, "a"] & S \end{tikzcd}$$

(5) A monoid with two generators σ and l

$$\begin{tikzcd} * \arrow[loop above, "\sigma"] \arrow[loop below, "l"] \end{tikzcd}$$

such that $\sigma^n l = l$ for $n = 1, 2, 3, ...,$ and $ll = l$.

(6) $$\begin{tikzcd} V \arrow[r, shift left, "s"] \arrow[r, shift right, "t"] & A \arrow[r, shift left=2, "\delta_0"] \arrow[r, "\delta_1"] \arrow[r, shift right=2, "\delta_2"] & T \end{tikzcd}$$

where $\delta_0 t = \delta_1 s$, $\delta_1 t = \delta_2 s$ and $\delta_2 t = \delta_0 s$

(7) $$\begin{tikzcd} V \arrow[r, shift left=2, "v_0"] \arrow[r, "v_1"] \arrow[r, shift right=2, "v_2"] & T \end{tikzcd}$$

(8) $$\begin{tikzcd} V \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & C \end{tikzcd}$$

where $f_0 t = f_1 t$ and $f_0 s = f_1 s$.

2 Representable $\mathbb{C}$-sets and Yoneda lemma

Let $\mathbb{C}$ be a category (whose objects and morphisms are thought of as generic figures and changes of figures). Every generic figure F may be considered itself as a container of generic figures or $\mathbb{C}$-set h_F whose F'-figures are the morphisms $F' \longrightarrow F \in \mathbb{C}$ with the operation of composing to the right as action:

$$\frac{F' \dashrightarrow h_F}{F' \longrightarrow F} \qquad \begin{tikzcd} & F' \arrow[r, dashed, "f"] & h_F \\ F'' \arrow[ur, "g"] \arrow[urr, dashed, "f.g = f \circ g"'] & & \end{tikzcd}$$

The $\mathbb{C}$-set h_F is called *representable* by F. Notice that every change of figures $F' \xrightarrow{f} F \in \mathbb{C}$ gives rise to a morphism $h_{F'} \xrightarrow{h_f} h_F$ of $\mathbb{C}$-sets defined by $h_f(g) = f \circ g$, where $F'' \xrightarrow{g} F' \in \mathbb{C}$ is an F''-figure of $h_{F'}$. To check that h_f is indeed a morphism, consider the diagram

$$\begin{tikzcd} & F'' \arrow[r, dashed, "g"] & h_{F'} \arrow[r, "h_f"] & h_F \\ F''' \arrow[ur, "h"] \arrow[urr, dashed, "g\circ f"'] & & & \end{tikzcd}$$

We have to show that $h_f(g).h = h_f(g \circ h)$, but this follows from the axiom of associativity of a category:

$$h_f(g).h = (f \circ g).h = (f \circ g) \circ h = f \circ (g \circ h) = h_f(g \circ h).$$

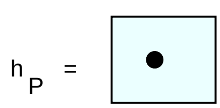
Notice that $h_{1_F} = 1_{h_F}$, namely the identity $h_F \longrightarrow h_F$.

2.1 Computation of representable $\mathbb{C}$-sets in the examples

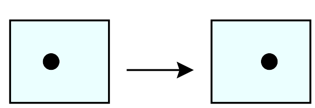
• *Sets*. In this case we have one representable $\mathbb{1}$-set: h_P. To find its graphical representation we extract the generic figures (points or P-figures):

$$\begin{array}{c} P \dashrightarrow h_P \\ \hline P \xrightarrow{1_P} P \\ \hline \text{one } P\text{-figure} \end{array}$$

There is only one trivial incidence relation. Hence, as a set



Clearly, there is only one morphism of $\mathbb{1}$-sets from h_P into itself, namely the identity h_{1_P}.



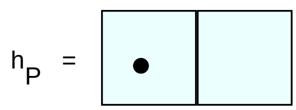
• *Bisets*. There are two representable 2-sets. To find their graphical representations we extract the generic figures, the P-figures and the S-figures:

$$\begin{array}{c} P \dashrightarrow h_P \\ \hline P \xrightarrow{1_P} P \\ \hline \text{one } P\text{-figure} \end{array} \qquad \begin{array}{c} S \dashrightarrow h_P \\ \hline S \longrightarrow P \\ \hline \text{no } S\text{-figure} \end{array}$$

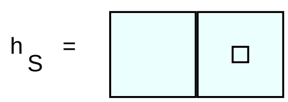
The incidence relations are trivial.

$$\begin{array}{c} S \dashrightarrow h_S \\ \hline S \xrightarrow{1_S} S \\ \hline \text{one } S\text{-figure} \end{array} \qquad \begin{array}{c} P \dashrightarrow h_S \\ \hline P \longrightarrow S \\ \hline \text{no } P\text{-figure} \end{array}$$

Hence, as sets



and

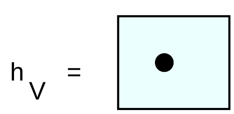


The only morphisms between representable 2-sets are the identities: h_{1_P} and h_{1_S}.

• *Bouquets*. There are two representable $(V \xrightarrow{v} L)$-sets: h_V and h_L. To represent them graphically we extract their generic figures:

$$\begin{array}{c} V \dashrightarrow h_V \\ \hline V \xrightarrow{1_V} V \\ \hline \text{one } V\text{-figure} \end{array} \qquad \begin{array}{c} L \dashrightarrow h_V \\ \hline L \longrightarrow V \\ \hline \text{no } L\text{-figure} \end{array}$$

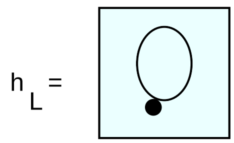
The incidence relations are trivial. Hence, as a bouquet:



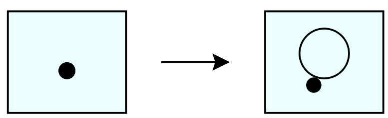
We proceed similarly with h_L:

$$\begin{array}{c} V \dashrightarrow h_L \\ \hline V \xrightarrow{v} L \\ \hline \text{one } V\text{-figure} \end{array} \qquad \begin{array}{c} L \dashrightarrow h_L \\ \hline L \xrightarrow{1_L} L \\ \hline \text{one } L\text{-figure} \end{array}$$

The only non trivial incidence relation is $1_L.v = v$, forced by the definition of the action as composition and the axioms for identities in a category. Hence, as a bouquet



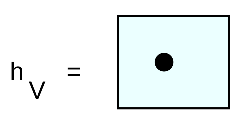
The only morphism between representable $(V \xrightarrow{v} A)$-sets (apart from the identities h_{1_V} and h_{1_L}) is the morphism $h_v : h_V \longrightarrow h_L$. This can be easily seen by looking at the graphical representation. For instance, h_v is the only morphism (of bouquets) from the bouquet with one vertex to the one with a loop:



• *Graphs*. There are two representable $\left(V \overset{s}{\underset{t}{\rightrightarrows}} A \right)$-sets: h_V and h_A. To represent them graphically we extract their generic figures:

$$\begin{array}{c} V \dashrightarrow h_V \\ \hline V \xrightarrow{1_V} V \\ \hline \text{one } V\text{-figure} \end{array} \qquad \begin{array}{c} A \dashrightarrow h_V \\ \hline A \longrightarrow V \\ \hline \text{no } A\text{-figure} \end{array}$$

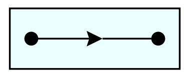
The incidence relations are trivial. Hence, as a graph:



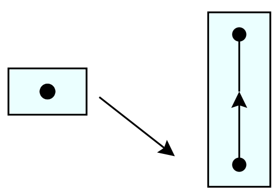
We proceed similarly with h_A:

$$\begin{array}{c} V \dashrightarrow h_A \\ \hline V \overset{s}{\underset{t}{\rightrightarrows}} A \\ \hline \text{two } V\text{-figures} \end{array} \qquad \begin{array}{c} A \dashrightarrow h_A \\ \hline A \xrightarrow{1_A} A \\ \hline \text{one } A\text{-figure} \end{array}$$

The non trivial incidence relations are $1_A.s = s$ and $1_A.t = t$, forced by the definition of the action as composition and the axioms for identities in a category. Hence, as a graph

$h_A =$ 

The only morphisms between representable ($V \overset{s}{\underset{t}{\rightrightarrows}} A$)-sets (apart from the identities h_{1_V} and h_{1_A}) are the morphisms $h_s : h_V \longrightarrow h_A$ and $h_t : h_V \longrightarrow h_A$. This can be easily seen by looking at the graphical representation. For instance, h_s is the morphism (of graphs) from the graph with one vertex into the graph with one arrow that sends the vertex into the source of the arrow:

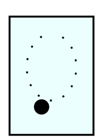


- *Rgraphs*. There are two representable ($V \begin{smallmatrix} \xrightarrow{s} \\ \xrightarrow{t} \\ \xleftarrow{l} \end{smallmatrix} A \circlearrowleft \sigma, \tau$)-sets: h_V and h_A.

To represent them graphically we extract their generic figures:

$$\begin{array}{c} V \dashrightarrow h_V \\ \hline V \xrightarrow{1_V} V \\ \hline \text{one } V\text{-figure} \end{array} \qquad \begin{array}{c} A \dashrightarrow h_V \\ \hline A \xrightarrow{l} V \\ \hline \text{one } A\text{-figure} \end{array}$$

Since there is only one vertex, the arrow must be a loop whose vertex is that vertex. Hence, as a reflexive graph:

$h_V =$ 

We proceed similarly with h_A:

$$\begin{array}{c} V \dashrightarrow h_A \\ \hline V \overset{s}{\underset{t}{\rightrightarrows}} A \\ \hline \text{two } V\text{-figures} \end{array} \qquad \begin{array}{c} A \dashrightarrow h_A \\ \hline A \longrightarrow A \\ \hline \text{three } A\text{-figures} \end{array}$$

The three A-figures are sl, tl and 1_A. Let us compute the incidence relations, namely, let us compute the source and the target of each of these A-figures. Remember that $ls = lt = 1_V$

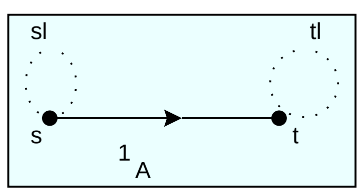
$$\begin{tikzcd} & A \arrow[r, dashed, "1_A"] & A \\ V \arrow[ur, "s"] \arrow[urr, dashed, "1_A.s=s"'] & & \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[r, dashed, "1_A"] & A \\ V \arrow[ur, "t"] \arrow[urr, dashed, "1_A.t=t"'] & & \end{tikzcd}$$

$$\begin{tikzcd} & A \arrow[r, dashed, "sl"] & A \\ V \arrow[ur, "s"] \arrow[urr, dashed, "sl.s=s"'] & & \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[r, dashed, "sl"] & A \\ V \arrow[ur, "t"] \arrow[urr, dashed, "sl.t=s"'] & & \end{tikzcd}$$

$$\begin{tikzcd} & A \arrow[r, dashed, "tl"] & A \\ V \arrow[ur, "s"] \arrow[urr, dashed, "tl.s=t"'] & & \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[r, dashed, "tl"] & A \\ V \arrow[ur, "t"] \arrow[urr, dashed, "tl.t=t"'] & & \end{tikzcd}$$

Hence, as a reflexive graph

h_A =



There are three morphisms between representable ($\begin{tikzcd} V \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & A \arrow[l, shift left=4, "l"] \arrow[loop, "\sigma"] \arrow[loop below, "\tau"] \end{tikzcd}$)-sets (apart from the identities h_{1_V} and h_{1_L}). There are two morphisms of reflexive graphs from h_V into h_A: one that sends the distinguished loop into the distinguished loop of the source and the other that sends it into the distinguished loop of the target. There is one morphism of reflexive graph that

sends h_A into h_V: it sends the distinguished loops and the arrow into the distinguished loop.

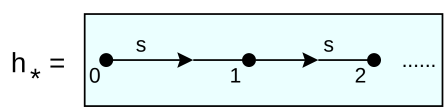
• *Esets*. There is only one representable $\mathbb{E}$-set: h_*. To represent it graphically we extract its generic figures:

$$\frac{\dfrac{* \dashrightarrow h_*}{* \longrightarrow *}}{\sigma^0, \sigma^1, \sigma^2, \ldots.}$$

The action on the generic figure σ^m by a morphism σ^n is given by $\sigma^m.\sigma^n = \sigma^{n+m}$:

$$\begin{tikzcd} & * \arrow[r, dashed, "\sigma^m"] & h_* \\ * \arrow[ur, "\sigma^n"] \arrow[urr, dashed, "\sigma^m.\sigma^n=\sigma^{m+n}"'] & & \end{tikzcd}$$

We find convenient to distinguished notationally the $*$-figures of h_* from the morphism $*\longrightarrow*$. Calling the first 0, 1, 2,... (rather than σ^0, σ^1, σ^2,...) we can rewrite the equation of the action as $m.\sigma^n = m + n$. This equation is equivalent to $m.\sigma = m+1 = S(m)$ where 'S' means 'successor'. Hence as an evolutive set $h_* = (\mathbb{N}, S)$. Graphically, the *generic chain*.



Notice that there are as many morphisms from the representable $\mathbb{E}$-sets h_* into itself as there are natural numbers: a morphism is completely determined by specifying the value at 0. Therefore there is a bijection between the morphisms of $h_*\longrightarrow h_*$ and the changes of figures from $*$ into itself.

`EXERCISE 2.1.1`

> If $\mathbb{C}$ is any of the categories of the exercise 1.2.2, compute the representable $\mathbb{C}$-Sets

2.2 Yoneda lemma

We have noticed that in all the examples there is a bijection between changes of generic figures $F \longrightarrow F'$ and morphisms of $\mathbb{C}$-sets $h_F \longrightarrow h_{F'}$. This is not a coincidence, but follows from the important:

Theorem 2.2.1 (Cayley, Grothendieck, Yoneda) *Let $\mathbb{C}$ be a category and let X be a $\mathbb{C}$-set. There is a canonical bijection between the F-figures of X and the morphisms of $\mathbb{C}$-sets between h_F and X:*

$$\frac{F \dashrightarrow X}{h_F \longrightarrow X}$$

Proof.
($\uparrow$): We send a morphism of $\mathbb{C}$-sets $\Phi : h_F \longrightarrow X$ into the F-figure $\Phi_F(1_F)$ of X.
($\downarrow$): Let

$$F \overset{\sigma}{\dashrightarrow} X$$

be an F-figure of X. We want to define $h_F \xrightarrow{\Phi} X$ such that $\Phi_F(1_F) = \sigma$. If such a Φ exists, $\Phi_{F'}(f) = \Phi_F(1_F).f = \sigma.f$ as can be seen by looking at the following diagram:

$$\begin{tikzcd} & F \arrow[r, dashed, "1_F"] & h_F \arrow[r, "\Phi"] & X \\ F' \arrow[ur, "f"] \arrow[urr, dashed, "f"'] & & & \end{tikzcd}$$

Thus, the definition of Φ is forced. Furthermore, the axioms for an action imply that Φ, thus defined, is indeed a morphism of $\mathbb{C}$-sets. Consider the diagram

$$\begin{tikzcd} & F' \arrow[r, dashed, "f"] & h_F \arrow[r, "\Phi"] & X \\ F'' \arrow[ur, "g"] \arrow[urr, dashed, "f.g"'] & & & \end{tikzcd}$$

We have to show that $\Phi_{F''}(f.g) = \Phi_{F'}(f).g$. But

$$\Phi_{F''}(f.g) = \sigma.(f.g) = (\sigma.f).g = \Phi_{F'}(f).g$$

Furthermore, these processes ($\uparrow$), ($\downarrow$) are inverse of each other. Starting from σ, we defined Φ precisely so that it is sent into σ by ($\uparrow$). On the other hand, let Φ be arbitrary and let $\sigma = \Phi_F(1_F)$. Then $\Phi_{F'}(f) = \Phi_F(1_F).f = \sigma.f$ (by looking at the first diagram of *Proof*). But, ($\downarrow$) sends σ into $\Phi^{'}$ defined by $\Phi^{'}_{F'}(f) = \sigma.f$. Then $\Phi = \Phi^{'}$. □

Corollary 2.2.2 (Yoneda lemma) *Let $\mathbb{C}$ be a category. There is a canonical bijection between morphisms $F \longrightarrow F' \in \mathbb{C}$ and morphisms of $\mathbb{C}$-sets $h_F \longrightarrow h_{F'}$:*

$$\frac{F \longrightarrow F'}{h_F \longrightarrow h_{F'}}$$

Proof.
Take $X = h_{F'}$ in the theorem. □

Both the theorem and the corollary will be referred to as *Yoneda lemma* in the sequel.

Notation. We use '$\overline{\sigma}$' to denote the morphism of $\mathbb{C}$-sets corresponding to the F-figure σ. Thus:

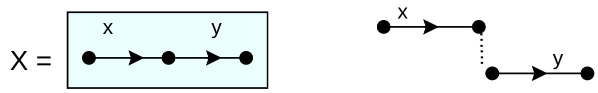
$$\frac{F \overset{\sigma}{\dashrightarrow} X}{h_F \xrightarrow{\overline{\sigma}} X}$$

Notice that $\overline{\sigma}(f) = \sigma.f$.

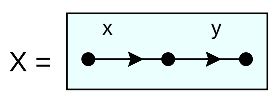
3 A $\mathbb{C}$-set as a container of generic figures

The reader has probably asked himself the question of what principle guided our choice of generic figures. The answer is not a simple matter and will be dealt with later on. For instance, why didn't we choose the loop as a further generic figure in graphs? Because a category different from *Graphs* would have been obtained! However, part of the answer must be that they suffice to 'build up' any $\mathbb{C}$-set. For graphs, this means that the *vertex* and the *arrow* suffice to build up any graph. But what do we mean by this? Take the concrete example of a graph with two arrows, one after the other, and three vertices. Then clearly this graph is obtained by taking two copies of the arrow and glueing them in such a way that the target of the first coincides

with the source of the second. The following picture speaks by itself:

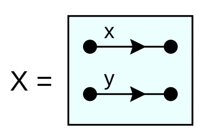


The aim of this section is to make this idea mathematically precise and to prove that every $\mathbb{C}$-set may be obtained by glueing (copies of) generic figures. Before doing this, however, we shall give a few examples of this glueing process in graphs, by listing the graph to be built and the scheme for glueing the (copies of the) generic figures to obtain it, side by side. This glueing scheme may be considered as a *blueprint* to build up the graph from generic figures. Thus, the previous example may be represented as:



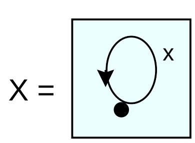
$$\begin{tikzcd} & A \arrow[dr, dashed, "x"] & \\ V \arrow[ur, "t"] \arrow[dr, "s"'] & & X \\ & A \arrow[ur, dashed, "y"'] & \end{tikzcd} \qquad xt = ys$$

A simpler graph is one having two arrows with no connection among them. This is obtained by taking two copies of the arrow with no glueing.

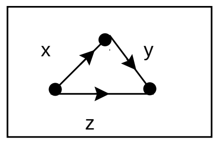


$$\begin{tikzcd} A \arrow[dr, dashed, "x"] & \\ & X \\ A \arrow[ur, dashed, "y"'] & \end{tikzcd} \qquad \text{no glueing}$$

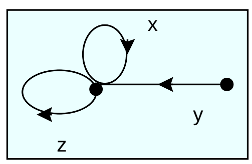
The following list may be extended at will. Notice that the glueing scheme is by no means unique. The reader is invited to find alternatives.



$$\begin{tikzcd} & A \arrow[dr, dashed, "x"] & \\ V \arrow[ur, "s"] \arrow[dr, "t"'] & & X \\ & A \arrow[ur, dashed, "x"'] & \end{tikzcd} \qquad xs = xt$$

X = 

$$\begin{tikzcd}
 & A \arrow[dr, dashed, "x"] & \\
V \arrow[ur, shift left, "s"] \arrow[ur, shift right, "t"'] \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] \arrow[dr, shift left, "s"] \arrow[dr, shift right, "t"'] & A \arrow[r, dashed, "y"] & X \\
 & A \arrow[ur, dashed, "z"'] &
\end{tikzcd}
\qquad
\begin{aligned} xs &= zs \\ xt &= ys \\ yt &= zt \end{aligned}$$

X = 

$$\begin{tikzcd}
 & A \arrow[dr, dashed, "x"] & \\
V \arrow[ur, shift left, "s"] \arrow[ur, shift right, "t"'] \arrow[r, "t"'] \arrow[dr, shift left, "s"] \arrow[dr, shift right, "t"'] & A \arrow[r, dashed, "y"] & X \\
 & A \arrow[ur, dashed, "z"'] &
\end{tikzcd}
\qquad
\begin{aligned} xs &= xt \\ xs &= yt \\ yt &= zt \\ zs &= zt \end{aligned}$$

By considering these examples, it should be clear that we may conclude in general

Proposition 3.0.3 *Every graph may be obtained by glueing (copies) of generic figures*

This proposition allows us to consider a graph as a *container* of generic figures (vertices and arrows) together with incidence relations. Before we prove that the same is true of arbitrary $\mathbb{C}$-sets, we give examples of glueing in the categories studied up to now.

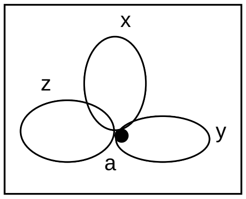
3.1 Glueing in the examples

• *Sets.* Every set is obtained by taking copies of the generic figure *point* with no glueing. Notice that we cannot glue two copies of the figure point without making them equal.

• *Bisets.* It should be clear that the situation is quite similar to sets and is left to the reader.

• *Bouquets.* We shall give one example, since the glueing process is quite

similar to, but simpler than graphs:

X = 

$$\begin{tikzcd}
& L \arrow[dr, dashed, "x"] & \\
V \arrow[ur, "v"] \arrow[r, "v"] \arrow[dr, "v"'] & L \arrow[r, dashed, "y"] & X \\
& L \arrow[ur, dashed, "z"'] &
\end{tikzcd}
\qquad xv = yv = zv = a$$

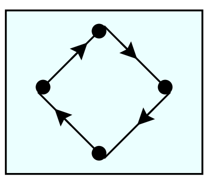
The reader will have no difficulty in concluding

Proposition 3.1.1 *Every bouquet may be obtained by glueing (copies of) generic figures*

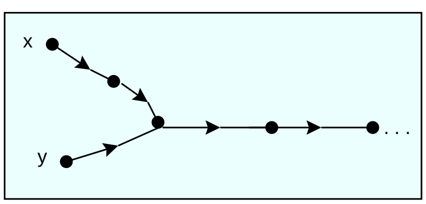
• *Graphs.* Already studied.

• *Rgraphs.* Left to the reader.

• *Esets.* The following are a few examples of how to obtain evolutive sets by glueing generic figures

X = 

$$\begin{tikzcd}
& * \arrow[dr, dashed, "x"] & \\
* \arrow[ur, "\sigma^4"] \arrow[dr, "1_*"'] & & X \\
& * \arrow[ur, dashed, "x"'] &
\end{tikzcd}
\qquad x\sigma^4 = x$$

X = 

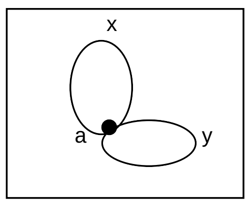
$$\begin{tikzcd}
& * \arrow[dr, dashed, "x"] & \\
* \arrow[ur, "\sigma^2"] \arrow[dr, "\sigma"'] & & X \\
& * \arrow[ur, dashed, "y"'] &
\end{tikzcd}
\qquad x\sigma^2 = y\sigma$$

Once again, the consideration of these examples allows us to conclude

Proposition 3.1.2 *Every evolutive set may be obtained by glueing (copies of) generic figures*

3.2 Glueing schemes as categories

It is clear that some glueing schemes or blueprints provide the right information to construct a given $\mathbb{C}$-set, whereas others are either sub-determined or over-determined. Take for instance

X = 

$$\begin{tikzcd} & L \arrow[dr, dashed, "x"] & \\ V \arrow[ur, "v"] \arrow[dr, "v"'] \arrow[rr, dashed, "a"] & & X \\ & L \arrow[ur, dashed, "y"'] & \end{tikzcd} \qquad xv = yv = a$$

The scheme on the right is a glueing scheme for X in the sense that X is the most adjusted bouquet with that scheme. We can make this precise by means of a universal property: assume that we have another bouquet Y with L-figures x', y' and such that $x'v = y'v$. Then there is a unique bouquet morphism $X \xrightarrow{\phi} Y$ such that $\phi(x) = x'$ and $\phi(y) = y'$. Graphically:

$$\begin{tikzcd} & L \arrow[dr, dashed, "x"'] \arrow[drr, dashed, "x'"] & & \\ V \arrow[ur, "v"] \arrow[dr, "v"'] & & X \arrow[r, "\phi"] & Y \\ & L \arrow[ur, dashed, "y"] \arrow[urr, dashed, "y'"'] & & \end{tikzcd}$$

To generalize, let us define the *category* $Fig(X)$ of generic figures and incidence relations of the $\mathbb{C}$-set X as follows: an object is an F-figure of X and a morphism $f : \tau \longrightarrow \sigma$ between an F'-figure of X and an F-figure of X is a change of figures $f : F' \longrightarrow F$ such that $\sigma.f = \tau$:

$$\begin{tikzcd} & F \arrow[r, dashed, "\sigma"] & X \\ F' \arrow[ur, "f"] \arrow[urr, dashed, "\tau=\sigma.f"'] & & \end{tikzcd}$$

Intuitively, $Fig(X)$ maybe thought of as the canonical (or maximal) glueing scheme or blueprint to build up X itself: all its F-figures are displayed and all the required glueing is indicated. Thus, we should expect that $Fig(X)$ is

a glueing scheme for X. Of course, we need to give a precise meaning to this notion of 'determination' which was defined only for bouquets.

We define a *glueing scheme* for X to be any subcategory S of $Fig(X)$ satisfying the following universal property: for every Y and every functor $\Phi : S \longrightarrow Fig(Y)$, which sends F-figures of X (in S) into F-figures of Y, there is a unique morphism $\phi : X \longrightarrow Y$ such that for every σ in S, $\phi(\sigma) = \Phi(\sigma)$.

The reader may check that this definition is equivalent to the universal property formulated in the example of bouquets.

We may now conclude

Theorem 3.2.1 *The category $Fig(X)$ is a glueing scheme for X.*

Proof. Let Y be any $\mathbb{C}$-set and let $\Phi : Fig(X) \longrightarrow Fig(Y)$ be a functor. We claim the existence of a unique morphism of $\mathbb{C}$-sets $\phi : X \longrightarrow Y$ such that for every F-figure σ of X, $\phi(\sigma) = \Phi(\sigma)$. By the last condition, ϕ is forced to be Φ (on objects). The fact that Φ is a functor, i.e., preserves incidence relations says precisely that ϕ respects the action. $\square$

The tautological nature of the proof tempts us to give a tautological statement of the theorem: any $\mathbb{C}$-set can be built up from (copies of) its figures, glued together according to its own glueing scheme.

`EXERCISE 3.2.1`

(1) Express $(\mathbb{Z}, \sigma)$, the evolutive set of the integers with successor as glueing of generic figures.

(2) Let

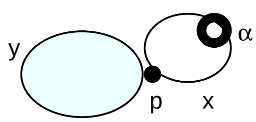
$$\begin{tikzcd}
& T \arrow[dr, "x"] & \\
V \arrow[ur, "v_1"] \arrow[dr, shift left=2, "v_1"] \arrow[dr, "v_2"] \arrow[dr, shift right=2, "v_3"'] & & X \\
& T \arrow[ur, "y"'] &
\end{tikzcd}$$

be the blueprint of a $\mathbb{C}$-set, where $\mathbb{C}$ is the category

$$\begin{tikzcd} V \arrow[r, shift left=3, "v_1"] \arrow[r, "v_2"] \arrow[r, shift right=3, "v_3"'] & T \end{tikzcd},$$

built a possible representation of the corresponding $\mathbb{C}$-set.

(3) Let



be a possible representation of a $\mathbb{C}$-set, where $\mathbb{C}$ is the category

$$A \xrightarrow{f} B \xrightarrow{g} C$$

and p is a A-figure, x, y are B-figures, α is a C-figure. Give a blue-print of this $\mathbb{C}$-set.

4 Operations on $\mathbb{C}$-sets

As mentioned in the Introduction $\mathbb{C}$-sets have several features of ordinary sets. In fact they have analogues of the singleton (terminal object), empty set (initial object) and are closed under several operations such as products, sums (or coproducts), inverse images, etc. which we proceed to describe in this section. Since these operations are defined by the usual universal properties of category theory, they are unique up to unique isomorphism which preserves the structural maps (of the universal properties).

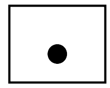
4.1 Finite limits and finite collimates

TERMINAL OBJECT

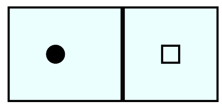
We define $\underline{1}$ to be a $\mathbb{C}$-set which has exactly one F-figure for each F, with the trivial action. It is a *terminal object* in the sense that for every $\mathbb{C}$-set X there is exactly one morphism of $\mathbb{C}$-sets $X \longrightarrow \underline{1}$.

Examples of terminal object.

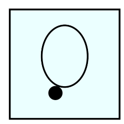
• *Sets.*



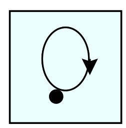
• *Bisets.*



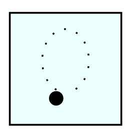
• *Bouquets*:



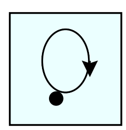
• *Graphs.*



• *Rgraphs.*



• *Esets.*



PRODUCTS

Given two $\mathbb{C}$-sets X and Y, we define $X \times Y$ to be the $\mathbb{C}$-set whose F-figures are couples (σ, τ), where σ is an F-figure of X and τ an F-figure of Y with the action $(\sigma, \tau).f = (\sigma ._X f, \tau ._Y f)$ for $f : F' \longrightarrow F$:

$$\frac{F \overset{(\sigma,\tau)}{\dashrightarrow} X \times Y}{F \overset{\sigma}{\dashrightarrow} X \;, \qquad F \overset{\tau}{\dashrightarrow} Y}
\qquad\qquad
\begin{tikzcd}
 & F \arrow[r, dashed, "{(\sigma,\tau)}"] & X \times Y \\
F' \arrow[ur, "f"] \arrow[urr, dashed, "{(\sigma,\tau).f=(\sigma._x f,\tau._y f)}"'] & &
\end{tikzcd}$$

It is easy to check that this definition makes $X \times Y$ into a $\mathbb{C}$-set. Furthermore there are obvious projection morphisms $X \times Y \xrightarrow{p_1} X$, which sends the F-figure (σ, τ) into the F-figure σ of X, and $X \times Y \xrightarrow{p_2} Y$, which sends the same couple into the F-figure τ of Y. With these morphisms, one can easily show that $X \times Y$ satisfies the universal property of a (categorical) product: $\forall Z \xrightarrow{f} X$, $\forall Z \xrightarrow{g} Y$ $\exists! Z \xrightarrow{h} X \times Y$ such that $p_1 \circ h = f$ and $p_2 \circ h = g$ ('$\exists!$' is interpreted as 'there is a unique'). This is usually represented by the following diagram which commute for the unique h:

$$\begin{tikzcd}
 & & X \\
Z \arrow[urr, "f"] \arrow[r, dashed, "h"] \arrow[drr, "g"'] & X \times Y \arrow[ur, "p_1"'] \arrow[dr, "p_2"] & \\
 & & Y
\end{tikzcd}$$

Notation: Notice that the symbol (dotted arrow) used for the existence of a unique arrow was used also for the extraction of a generic figure. They should not be confused.

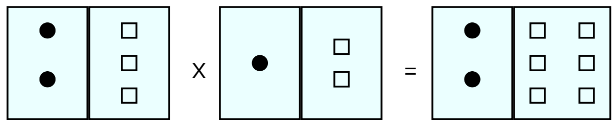
Although a product consists of an object and two projections we will often leave the projections out of the picture if the projections are clear from the context. We will do the same for the other operations.

As usual, we define $X^0 = \mathbb{1}$ and $X^n = \underbrace{X \times X \times ... \times X}_{\text{n times}}$.

The reader can prove that arbitrary products also exist, i.e., any family $(X_i)_i$ has a product satisfying the obvious universal property, which generalizes the binary case.

Examples of products.

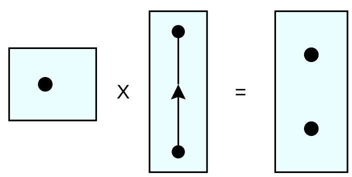
• *Bisets.*



• *Graphs.* Let us calculate the vertice-figures and the arrow-figures of $h_V \times h_A$

$$\begin{array}{c} V \dashrightarrow h_V \times h_A \\ \hline V \overset{1_V}{\dashrightarrow} h_V \quad , \quad V \dashrightarrow h_A \\ \hline \text{two vertices} \end{array} \qquad \begin{array}{c} A \dashrightarrow h_V \times h_A \\ \hline A \dashrightarrow h_V \quad , \quad A \overset{1_A}{\dashrightarrow} h_A \\ \hline \text{no arrows} \end{array}$$

We represent graphically this product as follows:



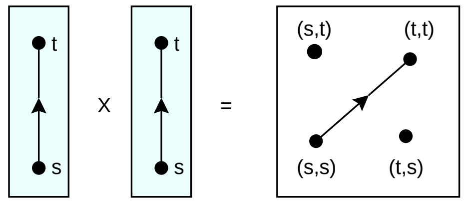
Let us now calculate the product $h_A \times h_A$

$$\begin{array}{c} V \dashrightarrow h_A \times h_A \\ \hline V \dashrightarrow h_A \quad , \quad V \dashrightarrow h_A \\ \hline \text{four vertices} \end{array} \qquad \begin{array}{c} A \dashrightarrow h_A \times h_A \\ \hline A \dashrightarrow h_A \quad , \quad A \dashrightarrow h_A \\ \hline \text{one arrow} \end{array}$$

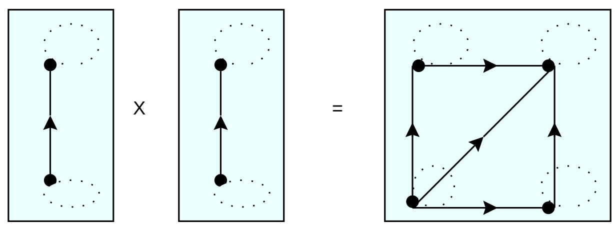
There are two V-figures of h_A: s and t. So let us call the vertices of $h_A \times h_A$: (s, s), (t, t), (s, t) and (t, s). We must calculate the incidence relations, for instance,

$$\begin{tikzcd} & A \arrow[r, dashed, "{(1_A,1_A)}"] & h_A \times h_A \\ V \arrow[ur, "s"] \arrow[urr, dashed, "{(s,s)}"'] & & \end{tikzcd}$$

This means that the source of the arrow is (s, s). In the same way we find that the target of the arrow is (t, t). We represent this product graphically as follows:



• *Rgraphs*. In the same way that we did for the graphs, we calculate the product $h_A \times h_A$ in the reflexive graphs and represent it graphically as follows:



EQUALIZERS

Let

$$X \underset{\Psi}{\overset{\Phi}{\rightrightarrows}} Y$$

be a diagram of $\mathbb{C}$-sets and $\mathbb{C}$-morphisms. We define $E(\Phi, \Psi)$ to be the sub $\mathbb{C}$-set of X whose F-figures σ are those F-figures of X such that $\Phi_F(\sigma) = \Psi_F(\sigma)$. Furthermore, we let $E(\Phi, \Psi) \overset{I}{\hookrightarrow} X$ be the inclusion morphism, i.e., the morphism whose F-component is the inclusion map from $E(\Phi, \Psi)_F$ into X_F.

It is easily checked that $E(\Phi, \Psi)$ is indeed a sub $\mathbb{C}$-set of X and that I is indeed a morphism. Furthermore, the diagram

$$E(\Phi, \Psi) \overset{I}{\hookrightarrow} X \underset{\Psi}{\overset{\Phi}{\rightrightarrows}} Y$$

is an *equalizer*.

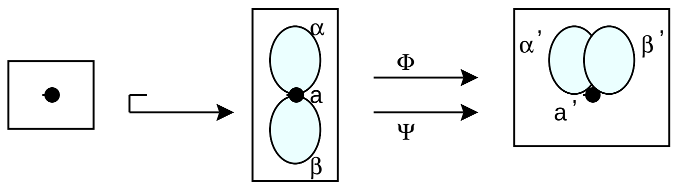
In other words, $\Phi \circ I = \Psi \circ I$ and the following universal property is satisfied: whenever $Z \xrightarrow{\Theta} X$ is a $\mathbb{C}$-morphisms such that $\Phi \circ \Theta = \Psi \circ \Theta$, there is a unique $\mathbb{C}$-morphism $Z \xrightarrow{\Theta'} E(\Phi, \Psi)$ making the following diagram commutative

$$
\begin{tikzcd}
E(\Phi,\Psi) \arrow[rr, hook, "I"] & & X \arrow[r, shift left, "\Phi"] \arrow[r, shift right, "\Psi"'] & Y \\
& Z \arrow[ul, "\Theta'"] \arrow[ur, "\Theta"'] & &
\end{tikzcd}
$$

in the sense that $I \circ \Theta' = \Theta$

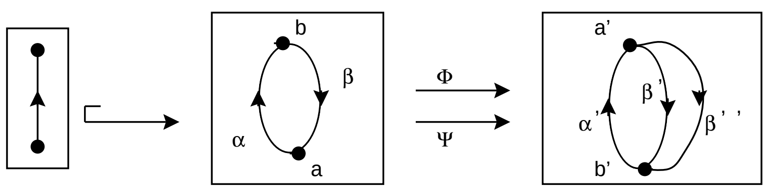
Examples of equalizers

- *Bouquets.*



$\Phi(\alpha) = \Phi(\beta) = \alpha'$, $\Psi(\alpha) = \Psi(\beta) = \beta'$ and $\Phi(a) = \Psi(a) = a'$.

- *Graphs.*



$\Phi(\alpha) = \alpha'$, $\Phi(\beta) = \beta'$, $\Phi(a) = a'$, $\Phi(b) = b'$, $\Psi(\alpha) = \alpha'$,$\Psi(\beta) = \beta'$, $\Psi(a) = a'$ and $\Psi(b) = b'$.

INVERSE IMAGES

Assume that

$$
\begin{tikzcd}
A \arrow[r, hook] & X \\
& Y \arrow[u, "\Phi"']
\end{tikzcd}
$$

is a diagram of $\mathbb{C}$-sets. We define the *inverse image* of A under Φ, $\Phi^*(A)$, to be the sub $\mathbb{C}$-set of Y whose F-figures are those of Y which are sent by Φ_F in an F-figure of A. In other words, the F-figures of $\Phi^*(A)$ are the

set-theoretical inverse image of $A(F)$ under Φ_F with the following action: if $F' \xrightarrow{f} F$ is a change of figure and $\sigma \in \Phi^*(A)(F)$, define $\sigma.f = \sigma._{Y} f$:

$$\frac{F \overset{\sigma}{\dashrightarrow} \Phi^*(A)}{\sigma \in Y(F) \quad , \quad \Phi_F(\sigma) \in A(F)} \qquad \begin{tikzcd} & F \arrow[r, dashed, "\sigma"] & \Phi^*(A) \\ F' \arrow[ur, "f"] \arrow[urr, dashed, "\sigma.f=\sigma._{Y} f"'] & & \end{tikzcd}$$

We have to check that $\sigma._{Y} f \in \Phi^*(A)(F')$. Since Φ respects the action,

$$\Phi_{F'}(\sigma._{Y} f) = \Phi_F(\sigma).f$$

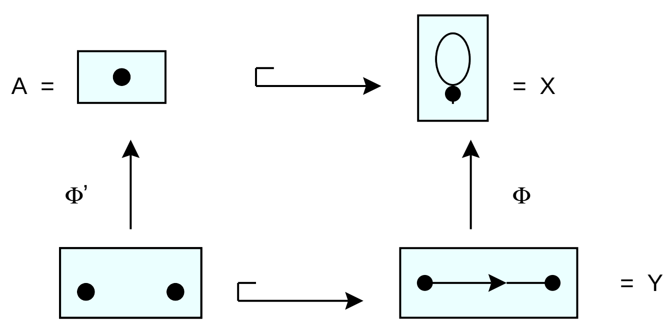
But $\Phi_F(\sigma) \in A(F)$ and A is a sub $\mathbb{C}$-set of X. Hence $\Phi_F(\sigma).f \in A(F')$, showing the conclusion. Thus, we have the set-theoretical commutative diagram:

$$\begin{tikzcd} A(F) \arrow[r, hook] & X(F) \\ \Phi^*(A)(F) \arrow[u, "\Phi'_F"] \arrow[r, hook] & Y(F) \arrow[u, "\Phi_F"'] \end{tikzcd}$$

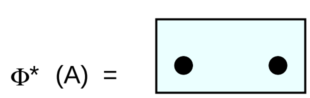
where $\Phi_F{}'$ is the restriction of Φ_F to $\Phi^*(A)(F)$.

The formulation of the universal property is left to the reader (see below the universal property for the pull backs).

Example of inverse images. The following is an inverse image in graphs.



Namely



Since the vertices of $\Phi^*(A)$ are those of Y sent by Φ into the vertices of A, i.e., all the vertices of Y. On the other hand $\Phi^*(A)$ has no arrows.

Products and inverse images are particular cases of pullbacks which we now describe.

PULLBACKS

Assume that

$$\begin{array}{ccc} A & \xrightarrow{\Psi} & X \\ & & \uparrow{\scriptstyle \Phi} \\ & & Y \end{array}$$

is a diagram of $\mathbb{C}$-sets. (We do not assume that A is a sub $\mathbb{C}$-set of X). We define $A \times_X Y$ as the sub $\mathbb{C}$-set of $A \times Y$ whose F-figures are couples (σ, τ) of F-figures of A and Y, respectively, and such that $\Psi_F(\sigma) = \Phi_F(\tau)$. Thus,

$$\frac{F \xrightarrow{(\sigma,\tau)} A \times_X Y}{\sigma \in A(F),\ \tau \in Y(F),\ \Psi_F(\sigma) = \Phi_F(\tau)}$$

$$\begin{tikzcd} & F \arrow[r, dashed, "{(\sigma,\tau)}"] & A \times_X Y \\ F' \arrow[ur, "f"] \arrow[urr, dashed, "{(\sigma,\tau).f=(\sigma._A f,\tau._Y f)}"'] & & \end{tikzcd}$$

Once again, we have to check that whenever (σ, τ) is an F-figure of $A \times_X Y$ and $F' \xrightarrow{f} F$ is a change of figure, then $(\sigma._A f, \tau._Y f)$ is an F'-figure of $A \times_X Y$. Details are similar to the previous case.

Thus, we require that the set-theoretical diagram

$$\begin{array}{ccc} A(F) & \xrightarrow{\Psi_F} & X(F) \\ {\scriptstyle p'_{1F}}\uparrow & & \uparrow{\scriptstyle \Phi_F} \\ A \times_X Y(F) & \xrightarrow[p'_{2F}]{} & Y(F) \end{array}$$

be a pullback. (Here p'_{1F} and p'_{2F} are the restrictions of the projections).

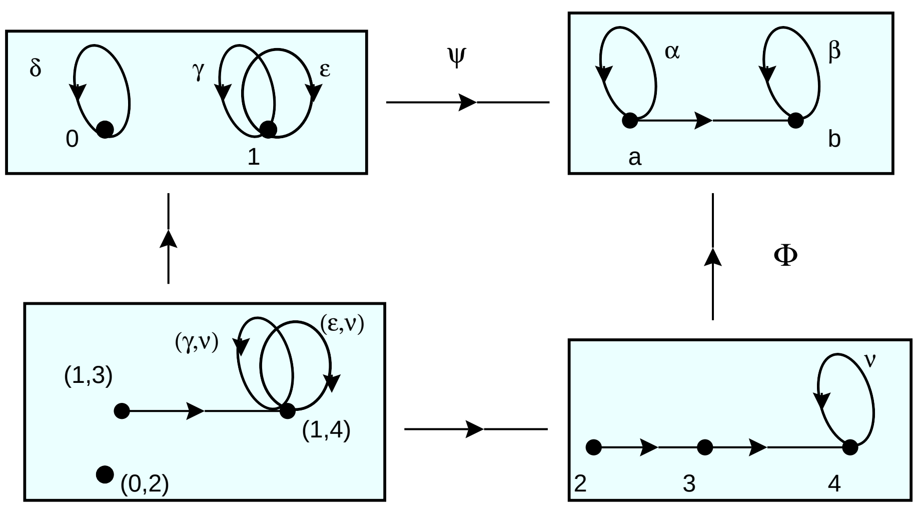
From the universal property of the pullback in sets, we derive the corresponding universal property of the pullback in $\mathbb{C}$-sets:

$$
\begin{tikzcd}
 & A \arrow[r, "\Psi"] & X \\
 & A \times_X Y \arrow[u, "p_1'"] \arrow[r, "p_2'"] & Y \arrow[u, "\Phi_F"'] \\
Z \arrow[uur] \arrow[ur, dashed] \arrow[urr] & &
\end{tikzcd}
$$

where p_1' and p_2' are the morphisms of $\mathbb{C}$-sets defined by $p_1'(F) = p_{1F}'$ and $p_2'(F) = p_{2F}'$.

As mentioned before, products are special cases of pullbacks. The reader can easily check that $X \times Y = X \times_{\mathbb{1}} Y$.

Examples of pullbacks: The following is a pullback in graphs for Ψ and Φ defined as follows $\Psi(0) = a$, $\Psi(1) = b$, $\Phi(2) = a$, $\Phi(3) = \Phi(4) = b$, the rest being forced:



INITIAL OBJECT

Define $\mathbb{0}$ to be the $\mathbb{C}$-set having no F-figures at all, with the trivial action. It is easy to check that $\mathbb{0}$ is an *initial object* in the sense that for every $\mathbb{C}$-set X there is a unique morphism of $\mathbb{C}$-sets: $\mathbb{0} \longrightarrow X$. On the other hand a $\mathbb{C}$-set having at least one F-figure is said to be *inhabited*

COPRODUCTS

If X and Y are $\mathbb{C}$-sets, we define the *coproduct* $X \bigsqcup Y$ (written sometimes '$X+Y$') to be the $\mathbb{C}$-set whose F-figures are the disjoint union of the F-figures of X and the F-figures of Y (e.g. $X(F) \bigsqcup Y(F) = X(F) \times \{0\} \cup Y(F) \times \{1\}$) with the following action: let $F' \xrightarrow{f} F$ be a change of figure and let (σ, i) be an F-figure of $X \bigsqcup Y$. There are two cases: if σ is an F-figure of X, then $i = 0$ and we let $(\sigma, 0).f = (\sigma ._X f, 0)$. If if σ is an F-figure of Y, then $i = 1$ and we let $(\sigma, 1).f = (\sigma ._Y f, 1)$. Further details are obvious. Notice that we have morphisms of $\mathbb{C}$-sets $i_1 : X \longrightarrow X \bigsqcup Y$ and $i_2 : Y \longrightarrow X \bigsqcup Y$. The first, for instance, takes an F-figure σ of X into the couple $(\sigma, 0)$. With these morphisms, the coproduct satisfies the following universal property:

$$\begin{tikzcd}
X \arrow[dr, "i_1"'] \arrow[drr] & & \\
& X \bigsqcup Y \arrow[r, dashed] & Z \\
Y \arrow[ur, "i_2"] \arrow[urr] & &
\end{tikzcd}$$

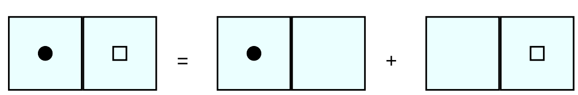
The universal property follows at once from the corresponding one for sets.

We define $nX = \underbrace{X \bigsqcup X \ ... \bigsqcup X}_{\text{n times}}$.

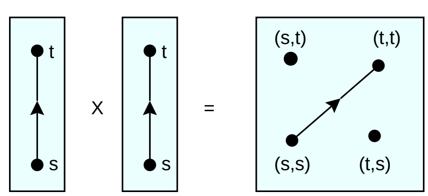
It is clear that any family $(X_i)_i$ has a coproduct. We leave the definition and the formulation of the universal property to the reader.

Examples of coproducts.

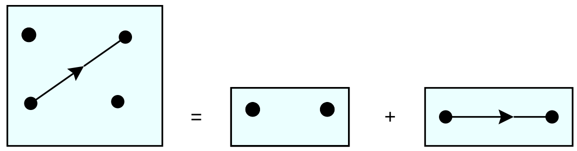
• *Bisets.* $\mathbb{1}$ is the coproduct of P and S



• *Graphs.* By looking at the following picture, we see that the product of the arrow by itself gives an arrow and two vertices.



So the product of an arrow by itself is the coproduct of an arrow and two vertices : $A \times A = 2V + A$. Graphically:



Similarly we can see that $V \times A = 2V$.

COEQUALIZERS

Let

$$X \overset{\Phi}{\underset{\Psi}{\rightrightarrows}} Y$$

be a diagram of $\mathbb{C}$-sets and $\mathbb{C}$-morphisms. We define a $\mathbb{C}$-set $C(\Phi, \Psi)$ and a morphism $Y \xrightarrow{Q} C(\Phi, \Psi)$ by stipulating that the diagram

$$X_F \overset{\Phi_F}{\underset{\Psi_F}{\rightrightarrows}} Y_F \xrightarrow{Q_F} C(\Phi, \Psi)_F$$

is the (canonical) coequalizer in Sets, i.e., $C(\Phi, \Psi)_F = Y_F / \sim_F$ where $\sim_F$ is the equivalence relation generated by the couples $\{(\Phi_F(\sigma), \Psi_F(\sigma)\}_{\sigma \in X_F}$ and Q_F is the map that sends $\tau \in Y_F$ into its equivalence class.

As before, it can be checked that $C(\Phi, \Psi)$ is a $\mathbb{C}$-set, Q a morphism and $Q \circ \Phi = Q \circ \Psi$. Furthermore the diagram

$$X \overset{\Phi}{\underset{\Psi}{\rightrightarrows}} Y \xrightarrow{Q} C(\Phi, \Psi)$$

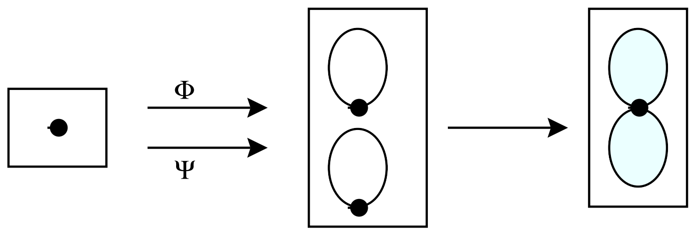
is a coequalizer in the sense that it satisfied the universal property: whenever $Y \xrightarrow{\Theta} Z$ is a $\mathbb{C}$-morphism such that $\Theta \circ \Phi = \Theta \circ \Psi$, there is a unique $\mathbb{C}$-morphism $C(\Phi, \Psi) \xrightarrow{\Theta'} Z$ making the following diagram commutative

$$\begin{tikzcd}
X \arrow[r, shift left, "\Phi"] \arrow[r, shift right, "\Psi"'] & Y \arrow[rr, "Q"] \arrow[dr, "\Theta"'] & & C(\Phi, \Psi) \arrow[dl, "\Theta'"] \\
& & Z &
\end{tikzcd}$$

in the sense that $\Theta' \circ Q = \Theta$.

Examples of coequalizers

- *Bouquets.*



EXERCISE 4.1.1

Let $X \xrightarrow{\Phi} Y$ be a morphism of $\mathbb{C}$-sets. Then Φ is an epimorphism iff

$$\begin{tikzcd} X \times_Y X \arrow[r, shift left, "p_1'"] \arrow[r, shift right, "p_2'"'] & X \arrow[r, "\Phi"] & Y \end{tikzcd}$$

is a coequalizer where p_1' ans p_2' are given by the pullback

$$\begin{tikzcd} X \arrow[r, "\Phi"] & X \\ X \times_Y X \arrow[u, "p_1'"] \arrow[r, "p_2'"'] & X \arrow[u, "\Phi"'] \end{tikzcd}$$

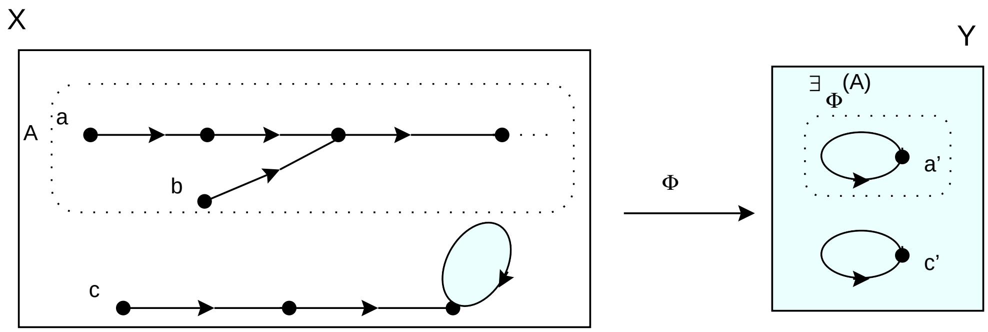
What would be the dual statement?

(DIRECT) IMAGE

Let $X \xrightarrow{\Phi} Y$ be a morphism of $\mathbb{C}$-sets and let $A \hookrightarrow X$ be a sub-$\mathbb{C}$-set of X. We define the *(direct) image* of A under Φ, $\exists_\Phi(A) \hookrightarrow Y$ to be the sub-$\mathbb{C}$-set of Y whose F-figures are those of Y coming via Φ_F from F-figures of A. In other words $\exists_\Phi(A)(F)$ is the ordinary set-theoretical (direct) image of $A(F) \subseteq X(F)$ under $X(F) \xrightarrow{\Phi_F} Y(F)$, i.e.,

$$\exists_\Phi(A) = \{\tau \in Y(F) | \tau = \Phi_F(\alpha) \text{ for some } \alpha \in A(F)\}$$

Example of (direct) image



EXERCISE 4.1.2

Show that every morphism $X \xrightarrow{\Phi} Y$ of $\mathbb{C}$-sets can be factored as follows

$$\begin{tikzcd} X \arrow[rr, "\Phi"] \arrow[dr, two heads, "\Psi"'] & & Y \\ & Z \arrow[ur, hook, "\Theta"'] & \end{tikzcd}$$

where Ψ is an epimorphism and Θ is a monomorphism.

Coproducts are particular cases of pushouts. We turn to these.

PUSHOUTS

Consider the following diagram of $\mathbb{C}$-sets:

$$\begin{tikzcd} X & \\ A \arrow[u, "\Phi"] \arrow[r, "\Psi"'] & Y \end{tikzcd}$$

We define $X \bigsqcup_A Y$ (also written as '$X +_A Y$') to be the $\mathbb{C}$-set whose F-figures $(X \bigsqcup_A Y)(F)$ are obtained from the following set-theoretical pushout diagram in sets:

$$\begin{tikzcd} X(F) \arrow[r, "i_{1F}"] & X(F) \bigsqcup_{A(F)} Y(F) \\ A(F) \arrow[u, "\Phi_F"] \arrow[r, "\Psi_F"'] & Y(F) \arrow[u, "i_{2F}"'] \end{tikzcd}$$

where i_{1F} and i_{2F} are the inclusions. The definition of the action will be postponed after we recall the construction of pushouts in sets.

Construction of pushouts in sets

We recall that the set-theoretical pushout

$$\begin{tikzcd} X \arrow[r, "i_1"] & X \bigsqcup_A Y \\ A \arrow[u, "\phi"] \arrow[r, "\psi"'] & Y \arrow[u, "i_2"'] \end{tikzcd}$$

is built in two steps: first we construct the disjoint union $X \bigsqcup Y$ and then we quotient this set by the smallest equivalence relation that contains the couple $(i_1(\phi(a)), i_2(\psi(a)))$ for every $a \in A$. Thus we have a map

$$[\,] : X \bigsqcup Y \longrightarrow X \bigsqcup_A Y$$

which sends an element into its equivalence class. Together with the maps

$$X \xrightarrow{[\,]\circ i_1} X \bigsqcup_A Y \ , \qquad Y \xrightarrow{[\,]\circ i_2} X \bigsqcup_A Y \ ,$$

that we abbreviate to i_1 and i_2, respectively, $X \bigsqcup_A Y$ satisfies the following universal property:

$$\begin{tikzcd} & & Z \\ X \arrow[r, "i_1"'] \arrow[urr] & X \bigsqcup_A Y \arrow[ur, dashed] & \\ A \arrow[u, "\phi"] \arrow[r, "\psi"'] & Y \arrow[u, "i_2"] \arrow[uur] & \end{tikzcd}$$

Returning to the $\mathbb{C}$-set $X \bigsqcup_A Y$, we define the action as follows:

$$[(\sigma, i)].f = [(\sigma, i).f] \ ,$$

where $i \in \{0, 1\}$. Of course, we have to show that this map is well-defined, i.e., does not depend on the choice of (σ, i) in its equivalence class. Thus

we have to show that for every change of figure $F \xrightarrow{f} F'$ if $[(\sigma, i)] = [(\tau, j)]$ then $[(\sigma, i).f] = [(\tau, j).f]$, where (σ, i) and (τ, j) are F-figures of $X \bigsqcup Y$ and $i, j \in \{0, 1\}$. Since the equivalence relation on $(X \bigsqcup Y)(F)$ is the smallest containing a certain set of couples, it is enough to check this for the generating couples of $(X \bigsqcup_A Y)(F)$, namely

$$((i_1)_F \Phi_F(\sigma), (i_2)_F \Psi_F(\sigma))$$

with σ an F-figure of A. But by definition of the equivalence relation on $(X \bigsqcup_A Y)(F')$, the couples

$$((i_1)_{F'} \Phi_{F'}(\sigma.f), (i_2)_{F'} \Psi_{F'}(\sigma.f))$$

belong to this equivalence relation and

$$\Phi_{F'}(\sigma.f) = \Phi_F(\sigma).f \ , \quad \Psi_{F'}(\sigma.f) = \Psi_F(\sigma).f \ .$$

The result follows.

Notice that we have morphisms of $\mathbb{C}$-sets:

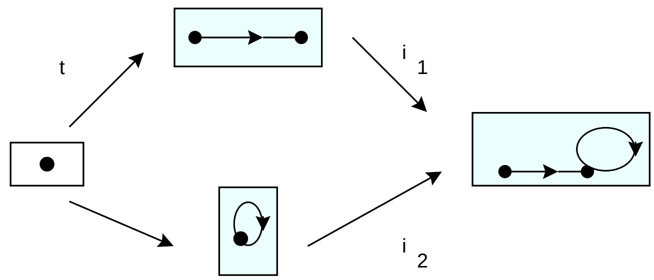
$$X \xrightarrow{i_1} X \bigsqcup_A Y \ \text{ and } \ Y \xrightarrow{i_2} X \bigsqcup_A Y$$

in terms of which we can state the following universal property which, incidentally, follows from the set-theoretical version:

$$\begin{tikzcd}
 & & Z \\
X \arrow[r, "i_1"'] \arrow[urr] & X \bigsqcup_A Y \arrow[ur, dashed] & \\
A \arrow[u, "\Phi"] \arrow[r, "\Psi"'] & Y \arrow[u, "i_2"'] \arrow[uur] &
\end{tikzcd}$$

Further details are left to the reader.

Example of pushouts. The following is a pushout in graphs:



This example may be thought of as glueing figures, although not generic ones: we glue the loop to the target of the arrow.

4.2 Limits and colimits

Terminal objects, products and pullbacks in a category $\mathbb{A}$ are particular cases of limits of functors, while initial objects, coproducts and pushouts are instances of colimits of functors.

COLIMITS

Let $\mathbb{I}$ be a category (thought of as the category of 'indices') and let

$$F : \mathbb{I} \longrightarrow \mathbb{A}$$

be a functor.

We define a *colimit* of F to be an object A of $\mathbb{A}$ together with a family $(F(i) \xrightarrow{\eta_i} A)_{i \in \mathbb{I}}$ of morphisms of $\mathbb{A}$ satisfying the following properties:

(1) For every $i \xrightarrow{\alpha} j \in \mathbb{I}$, the diagram

$$\begin{tikzcd} F(i) \arrow[dd, "F(\alpha)"'] \arrow[dr, "\eta_i"] & \\ & A \\ F(j) \arrow[ur, "\eta_j"'] & \end{tikzcd}$$

is commutative, i.e., $\eta_j \circ F(\alpha) = \eta_i$.

(2) Whenever $(F(i) \xrightarrow{\phi_i} B)_{i \in \mathbb{I}}$ is any family (of morphisms of $\mathbb{A}$) with the same property, i.e., $\phi_i = \phi_j \circ F(\alpha)$, there is a unique morphism $A \xrightarrow{\Phi} B$ such that $\Phi \circ \eta_i = \phi_i$ for all i. In terms of diagrams:

$$
\begin{tikzcd}
F(i) \arrow[dd, "F(\alpha)"'] \arrow[dr, "\eta_i"] \arrow[drr, "\phi_i"] & & \\
& A \arrow[r, dashed, "\Phi"] & B \\
F(j) \arrow[ur, "\eta_j"] \arrow[urr, "\phi_j"'] & &
\end{tikzcd}
$$

Remark 4.2.1 A family $(F(i) \xrightarrow{\phi_i} B)_{i \in \mathbb{I}}$ such that for every $i \xrightarrow{\alpha} j \in \mathbb{I}$

$$
\begin{tikzcd}
F(i) \arrow[dd, "F(\alpha)"'] \arrow[dr, "\phi_i"] & \\
& B \\
F(j) \arrow[ur, "\phi_j"] &
\end{tikzcd}
$$

is commutative is called a *cocone* for F. The cocone defining a colimit is called a *colimiting cocone*.

This notion becomes more transparent when reformulated in terms of constant functors.

Let $\Delta : \mathbb{A} \longrightarrow \mathbb{A}^{\mathbb{I}}$ be the *diagonal* functor such that, for each $A \in \mathbb{A}$,

$$\Delta A : \mathbb{I} \longrightarrow \mathbb{A}$$

is the *constant* functor defined by

$$\begin{cases} \Delta A(i) = A \\ \Delta A(i \xrightarrow{\alpha} j) = A \xrightarrow{1_A} A \end{cases}$$

Although F is not constant (in general), we may ask whether there is an A such that ΔA is the constant functor 'closest' to F in the sense that

there is a natural transformation $F \xrightarrow{\eta} \Delta A$ which is universal with respect to these transformations: for every $F \xrightarrow{\theta} \Delta B$ there is a unique $A \xrightarrow{\Phi} B$ such that the diagram

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr, "\theta"'] & \Delta A \arrow[d, "\Delta\Phi"] \\ & \Delta B \end{tikzcd}$$

is commutative.

By spelling this universal property out, it turns out that the natural transformation $(F(i) \xrightarrow{\eta_i} A)_{i \in \mathbb{I}}$ is a colimit of F, since a natural transformation $F \longrightarrow \Delta B$ is a cocone for F.

It is easy to see that colimits are unique up to a unique isomorphism: if $F \xrightarrow{\eta} \Delta(A)$ and $F \xrightarrow{\theta} \Delta(B)$ are colimits, then there is a unique isomorphism $\Phi : A \xrightarrow{\sim} B$ such that $\Delta(\Phi) \circ \eta = \theta$. Thus, we may talk about *the* colimit of a functor.

We will return to this way of looking at colimits in 8.2.4.

Construction of colimits in sets

We recall that the colimit of a functor

$$\mathbb{I} \xrightarrow{F} Set$$

is also buit in two steps: first we construct the disjoint union as the set

$$\bigsqcup_i F(i) = \{(i, a) | i \in \mathbb{I}, a \in F(i)\}$$

with inclusion maps $\eta_i : F(i) \longrightarrow \bigsqcup_i F(i)$, defined by $\eta_i(a) = (i, a)$, and then we quotient this set by the smallest equivalence relation $(\sim)$ such that $(i, a) \sim (j, F(\alpha)(a))$ for $i \xrightarrow{\alpha} j$. Thus we have a map

$$[\,] : \bigsqcup_i F(i) \longrightarrow \bigsqcup_i F(i)/\sim$$

which sends an element into its equivalence class. Together with the maps

$$F(i) \xrightarrow{[\,]_i} \bigsqcup_i F(i)/\sim, \qquad F(j) \xrightarrow{[\,]_j} \bigsqcup_i F(i)/\sim,$$

obtained by composing $[\,]$ with the inclusions,

$\bigsqcup_i F(i)/\sim$ satisfies the following universal property:

$$\begin{tikzcd}
F(i) \arrow[dd, "F(\alpha)"'] \arrow[dr, "{[\,]_i}"'] \arrow[drr, "\phi_i"] & & \\
& \bigsqcup_i F(i)/\sim \arrow[r, dashed, "\Phi"] & S \\
F(j) \arrow[ur, "{[\,]_j}"] \arrow[urr, "\phi_j"'] & &
\end{tikzcd}$$

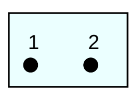
Examples of colimits

• *Initial object*
Take $\mathbb{I}$ to be the empty category $\emptyset$ and $F : \emptyset \longrightarrow \mathbb{A}$ the only functor , i.e., the empty functor. Then $colimF$ is the initial object.

• *Coproducts*
We wish to express the coproduct diagram

$$\begin{tikzcd}
X \arrow[dr, "i_1"] & \\
& X \bigsqcup Y \\
Y \arrow[ur, "i_2"'] &
\end{tikzcd}$$

as a colimit. Take $\mathbb{I}$ to be a set with two elements $1, 2$ (seen as a category)



and let F be the functor defined by $F(1) = X$ and $F(2) = Y$. Then the colimit of F is the family consisting of the two morphisms:

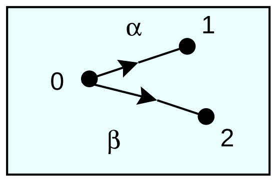
$$\eta_1 = i_1 : X \longrightarrow X \bigsqcup Y \ , \ \eta_2 = i_2 : Y \longrightarrow X \bigsqcup Y$$

Notice that the universal property (2) reduces to that of a coproduct.

• *Pushouts*
We want to express the pushout diagram

$$\begin{tikzcd}
 & X \arrow[dr, "i_1"] & \\
A \arrow[ur, "f"] \arrow[dr, "g"'] & & X\bigsqcup Y \\
 & Y \arrow[ur, "i_2"'] &
\end{tikzcd}$$

as a colimit. Take $\mathbb{I}$ to be the category



Define F to be the functor such that $F(0) = A$, $F(1) = X$, $F(2) = Y$ on objects, and $F(\alpha) = f$, $F(\beta) = g$ on morphisms. Then the colimit of F is the family consisting of three morphisms

$$\eta_0 = i_1 \circ f = i_2 \circ g : A \longrightarrow X \bigsqcup_A Y,\ \eta_1 = i_1 : X \longrightarrow X \bigsqcup_A Y,\ \eta_2 = i_2 : Y \longrightarrow X \bigsqcup_A Y.$$

Once again, the universal property (2) reduces to that of a pushout.

• *Glueing as colimits*
The glueing process studied in section 3 may be formulated in terms of colimits. Theorem 3.2.1 says that $Fig(X)$ is a glueing scheme for X. This raises the question whether there is an operation that acting on $Fig(X)$ gives X as a result. The answer can be given in terms of colimits.

Let $U_X : Fig(X) \longrightarrow \mathbb{C}$-sets be the forgetful functor which sends the F-figure σ into h_F.

Theorem 4.2.2 *Every $\mathbb{C}$-set X is the colimit of the forgetful functor*

$$U_X : Fig(X) \longrightarrow \mathbb{C} - \text{sets}.$$

Proof. To simplify, we shall identify (by Yoneda) F and h_F. Thus, the F-figure

$$(F \overset{\sigma}{\dashrightarrow} X)$$

will be identified with the morphism of $\mathbb{C}$-sets $\overline{\sigma} : h_F \xrightarrow{\sigma} X$, which will be written simply as '$F \xrightarrow{\sigma} X$'.

By definition of colimit we have to define for every $F \xrightarrow{\sigma} X$ a morphism of $\mathbb{C}$-sets

$$\eta_\sigma : U_X(F \xrightarrow{\sigma} X) \longrightarrow X.$$

such that $\forall f : (F' \xrightarrow{\sigma'} X) \longrightarrow (F \xrightarrow{\sigma} X) \in Fig(X)$ the following diagram is commutative

$$\begin{tikzcd}
U_X(F \xrightarrow{\sigma} X) \arrow[dr, "\eta_\sigma"] & \\
& X \\
U_X(F' \xrightarrow{\sigma'} X) \arrow[uu, "f"] \arrow[ur, "\eta_{\sigma'}"'] &
\end{tikzcd}$$

By taking $\eta_\sigma = \sigma$, the commutativity of the diagram says precisely that f is a morphism of $Fig(X)$. Thus it is trivially verified.

Let $(U_X(F \xrightarrow{\sigma} X) \xrightarrow{\Phi_\sigma} Y)_\sigma$ be a family such that

$$\forall f : (F' \xrightarrow{\sigma'} X) \longrightarrow (F \xrightarrow{\sigma} X) \in Fig(X)$$

the following diagram is commutative

$$\begin{tikzcd}
U_X(F \xrightarrow{\sigma} X) \arrow[dr, "\Phi\sigma"] & \\
& Y \qquad (*) \\
U_X(F' \xrightarrow{\sigma'} X) \arrow[uu, "f"] \arrow[ur, "\Phi\sigma'"'] &
\end{tikzcd}$$

Define $X \xrightarrow{\phi} Y$ by sending the F-figure σ of X into Φ_σ (a definition that is forced). To check that ϕ respects the action is the same as to check that $\Phi_\sigma f = \Phi_{\sigma'}$, which is precisely (*).□

• $\bigvee$ *in posets*
Take $\mathbb{I}$ to be the set I. A family (a_i) may be identified with the functor $f : I \longrightarrow P$, where P is a poset, such that $f(i) = a_i$. Then the family $(\eta_i)_{i \in I}$ of the inequalities $a_i \le \bigvee_{i \in I} a_i$ is the colimit of F. From this point of view, colimits generalize suprema (sups). Similarly, limits (see below) generalize infima (infs).

LIMITS

Limits may be defined as colimits in the dual category $\mathbb{A}^{op}$.

Let us spell this out in detail. Let $F : \mathbb{I} \longrightarrow \mathbb{A}$ be a functor. We define a *limit* of F to be a family $(A \xrightarrow{\epsilon_i} F(i))_{i \in \mathbb{I}}$ satisfying the two conditions:

(1) For every $i \xrightarrow{\alpha} j \in \mathbb{I}$ the diagram

$$\begin{tikzcd} & F(i) \arrow[dd, "F(\alpha)"] \\ A \arrow[ur, "\epsilon_i"] \arrow[dr, "\epsilon_j"'] & \\ & F(j) \end{tikzcd}$$

is commutative, i.e., $F(\alpha) \circ \epsilon_i = \epsilon_j$.

(2) Whenever $(B \xrightarrow{\psi_i} F(i))_{i \in \mathbb{I}}$ is any family with the same property (i.e., $F(\alpha) \circ \psi_i = \psi_j$), there is a unique map $B \xrightarrow{\Psi} A$ such that $\epsilon_i \circ \Psi = \psi_i$. In diagrammatical form:

$$\begin{tikzcd} & & F(i) \arrow[dd, "F(\alpha)"] \\ B \arrow[urr, "\psi_i"] \arrow[r, dashed, "\Psi"] \arrow[drr, "\psi_j"'] & A \arrow[ur, "\epsilon_i"] \arrow[dr, "\epsilon_j"'] & \\ & & F(j) \end{tikzcd}$$

We leave to the reader the tasks of defining cones, limiting cones and of reformulating the notion of limit both in terms of these notions and in terms of constant functors.

EXERCISE 4.2.1

> Reformulate the notions of terminal objects, products and pullbacks in terms of limits.

4.3 Exponentiation

The basic property of exponentials in sets is the existence of a bijection

$$\frac{Z \xrightarrow{f} Y^X}{Z \times X \xrightarrow{g} Y} \qquad g = f^{\cup}\ ,\ f = g^{\cap}$$

where $f^{\cup}(z,x) = f(z)(x)$ and $g^{\cap}(z)(x) = g(z,x)$. Notice that in the definition of $f^{\cup}$ we are required to evaluate the function $f(z)$ at x. But evaluation of a function at a point is itself a function, namely the function

$$Y^X \times X \xrightarrow{e} Y$$

corresponding to 1_{Y^X} in the above bijection, since $(1_{Y^X})^{\cup}(f,x) = 1_{Y^X}(f)(x) = f(x)$. Diagramatically, we may represent the passage from f to $f^{\cup}$ as follows

$$\frac{\dfrac{Z \xrightarrow{f} Y^X}{Z \times X \xrightarrow{f \times 1_X} Y^X \times X\ ,\ Y^X \times X \xrightarrow{e} Y}}{Z \times X \xrightarrow{e\,\circ(f \times 1_X)} Y}$$

We can reformulate the existence of the above bijection by defining the *exponential* of Y and X to be the set Y^X together with a map $Y^X \times X \xrightarrow{e} Y$ (the evaluation) such that the process of going from f to $f^{\cup} = e \circ f \times 1_X$ (in the preceding diagram) is a bijection.

Since this reformulation makes sense in any category with products, we take it as the definition of the exponential in the category of $\mathbb{C}$-sets. Thus, we are required to define two things: a $\mathbb{C}$-set Y^X and an evaluation map $Y^X \times X \xrightarrow{e} Y$ in such a way that the passage from f to $f^{\cup}$ is a bijection.

To define the F-figures of Y^X we recall that Yoneda gives a bijection

$$\frac{F \dashrightarrow Y^X}{h_F \longrightarrow Y^X}$$

From the basic property of exponentials, these maps should be in a one-to-one correspondence with the maps

$$h_F \times X \longrightarrow Y$$

It seems natural, therefore, to define the F-figures of Y^X to be the morphisms of $\mathbb{C}$-sets $h_F \times X \longrightarrow Y$ with the action $\sigma.f = \sigma \circ (h_f \times 1_X)$:

$$\frac{F \overset{\sigma}{\dashrightarrow} Y^X}{h_F \times X \xrightarrow{\sigma} Y} \qquad \begin{tikzcd} & h_F \times X \arrow[r, "\sigma"] & Y \\ h_{F'} \times X \arrow[ur, "h_f \times 1_X"] \arrow[urr, "\sigma.f = \sigma \circ (h_f \times 1_X)"'] & & \end{tikzcd}$$

To define the evaluation $Y^X \times X \xrightarrow{e} Y$ let us identify h_F with F and h_f with f (as done on several occasions). Consider an F-figure of $Y^X \times X$, i.e., a couple (ϕ, σ), with $F \times X \xrightarrow{\phi} Y$ and $F \xrightarrow{\sigma} X$. Define $e(\phi, \sigma) = \phi(1_F, \sigma)$:

$$\begin{tikzcd} & F \times X \arrow[r, "\phi"] & Y \\ F \arrow[ur, dashed, "(1_F,\sigma)"] \arrow[urr, dashed, "e(\phi,\sigma)=\phi(1_F,\sigma)"'] & & \end{tikzcd}$$

To check that e respects the action, let $F' \xrightarrow{f} F$ be a change of figure. Consider the diagram

$$\begin{tikzcd} & F \arrow[r, dashed, "(\phi,\sigma)"] & Y^X \times X \arrow[r, "e"] & Y \\ F' \arrow[ur, "f"] \arrow[urr, dashed, "(\phi.f,\sigma.f)"'] & & & \end{tikzcd}$$

We claim that $e((\phi, \sigma).f) = e(\phi, \sigma).f$ Indeed,

$$e((\phi, \sigma).f) \quad = \quad e(\phi.f, \sigma.f)$$

$$
\begin{aligned}
&= \phi.f(1_{F'}, \sigma.f) && \text{(by definition of } e\text{)} \\
&= (\phi \circ (f \times 1_X))(1_{F'}, \sigma.f) && \text{(action in exponentials)} \\
&= \phi((f \times 1_X)(1_{F'}, \sigma.f)) \\
&= \phi(f, \sigma.f) \\
&= \phi((1_F, \sigma).f) \\
&= \phi(1_F, \sigma).f && \text{(since } \phi \text{ is a morphism)} \\
&= e(\phi, \sigma).f && \text{(by definition of } e\text{)}
\end{aligned}
$$

The following result states that the $\mathbb{C}$-set Y^X together with the evaluation $Y^X \times X \xrightarrow{e} Y$ is an exponential:

Theorem 4.3.1 *The passage from a map f in the top to $f^{\cup} = e \circ (f \times 1_X)$ in the bottom induces a bijection*

$$
\frac{Z \longrightarrow Y^X}{Z \times X \longrightarrow Y}
$$

Proof.
The proof will be given in section 8. However the reader is encouraged to prove it at this point.

The exponential Y^X, together with the evaluation map $Y^X \times X \xrightarrow{e} Y$ satisfies the following universal property which is but another way of stating the above bijection: for every $Z \times X \xrightarrow{f} Y$ there is a unique $Z \xrightarrow{g} Y^X$ such that the diagram

$$
\begin{tikzcd}
Z \times X \arrow[rr, "g \times 1_X"] \arrow[dr, "f"'] & & Y^X \times X \arrow[dl, "e"] \\
& Y &
\end{tikzcd}
$$

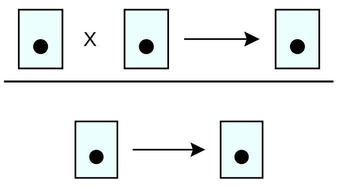
is commutative.

Exponentials in some examples

• *Bouquets*. Let us compute V^V by looking at its V-figures, L-figures and

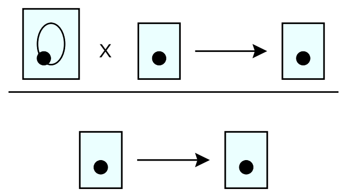
the incidence relations:

$$\frac{\dfrac{V \longrightarrow V^V}{V \times V \longrightarrow V}}{}$$



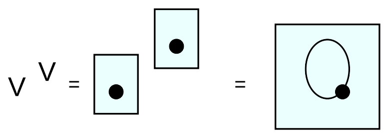
1 vertex

$$\frac{\dfrac{L \longrightarrow V^V}{L \times V \longrightarrow V}}{}$$



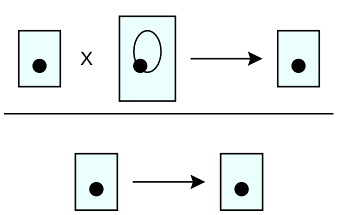
1 loop

Hence



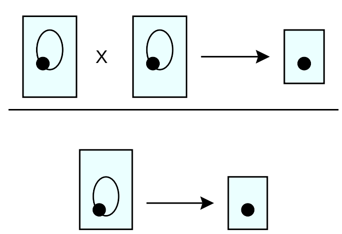
Let us turn to the computation of V^L:

$$\frac{\dfrac{V \longrightarrow V^L}{V \times L \longrightarrow V}}{}$$



1 vertex

$$\frac{\dfrac{L \longrightarrow V^L}{L \times L \longrightarrow V}}{}$$



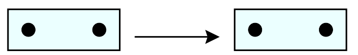
no loops

Hence $V^L = V$

A similar computation gives $L^V = L^L = L$

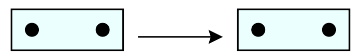
As a last example of exponentials in bouquets, let us compute $(2V)^{(2V)}$:

$$\frac{\dfrac{V \longrightarrow (2V)^{(2V)}}{V \times 2V \longrightarrow 2V}}{}$$



4 vertices

$$\frac{\dfrac{L \longrightarrow (2V)^{(2V)}}{L \times 2V \longrightarrow 2V}}{}$$

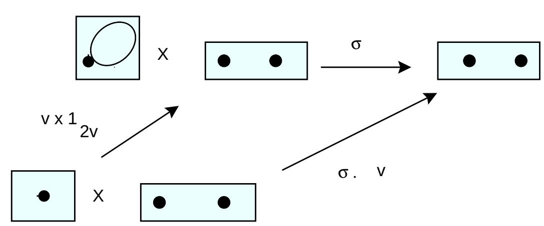


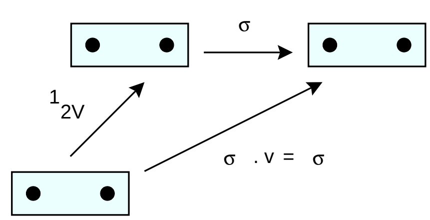
4 loops

Incidence relations:

$$\begin{tikzcd} & L \arrow[r, dashed, "\sigma"] & (2V)^{(2V)} \\ V \arrow[ur, "v"] \arrow[urr, dashed, "\sigma.v"'] & & \end{tikzcd}$$

$$\begin{tikzcd} & L \times (2V) \arrow[r, "\sigma"] & (2V) \\ V \times 2V \arrow[ur, "v\times 1_{2V}"] \arrow[urr, "\sigma.v=\sigma\circ(v\times 1_{2V})"'] & & \end{tikzcd}$$

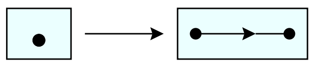


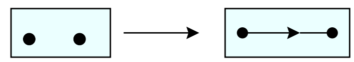


Hence $(2V)^{(2V)} = 4L = L + L + L + L$

• *Graphs*. As a first example, we shall compute A^V:

$$\begin{array}{c} V \longrightarrow A^V \\ \hline V \times V \longrightarrow A \\ \hline V \longrightarrow A \\ \hline \end{array} \qquad\qquad \begin{array}{c} A \longrightarrow A^V \\ \hline A \times V \longrightarrow A \\ \hline 2V \longrightarrow A \\ \hline \end{array}$$





2 vertices　　　　4 arrows

Let the vertices and the arrows be s, t and

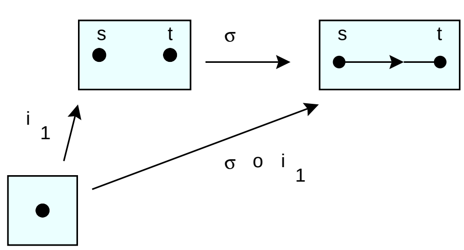
$$\sigma_1 = \begin{cases} s \mapsto s \\ t \mapsto s \end{cases} \qquad \sigma_2 = \begin{cases} s \mapsto s \\ t \mapsto t \end{cases} \qquad \sigma_3 = \begin{cases} s \mapsto t \\ t \mapsto s \end{cases} \qquad \sigma_4 = \begin{cases} s \mapsto t \\ t \mapsto t \end{cases}$$

To compute the source (and similarly the target) of the arrows we proceed as follows:

$$\begin{tikzcd} & A \arrow[r, dashed, "\sigma"] & A^V \\ V \arrow[ur, "s"] \arrow[urr, dashed, "\sigma.s"'] & & \end{tikzcd}$$

$$\begin{tikzcd} & A \times V \arrow[r, "\sigma"] & A \\ V \times V \arrow[ur, "s\times 1_V"] \arrow[urr, "\sigma.s=\sigma\circ(s\times 1_V)"'] & & \end{tikzcd}$$

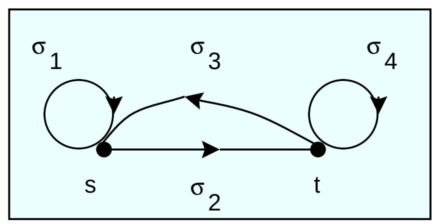
$$\begin{tikzcd} & V + V \arrow[r, "\sigma"] & A \\ V \arrow[ur, "i_1"] \arrow[urr, "\sigma.s=\sigma\circ i_1"'] & & \end{tikzcd}$$



The incidence relations are thus:

$$\begin{array}{cccc} \sigma_1 s = s & \sigma_2 s = s & \sigma_3 s = t & \sigma_4 s = t \\ \sigma_1 t = s & \sigma_2 t = t & \sigma_3 t = s & \sigma_4 t = t \end{array}$$

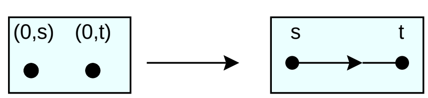
Hence

$A^V =$ 

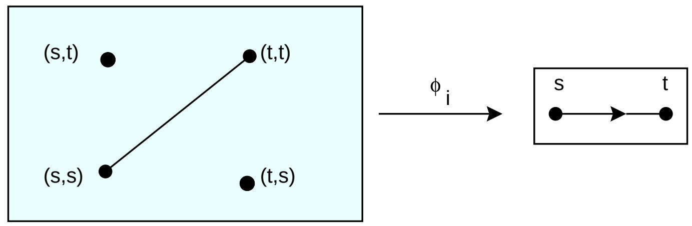
As the final example of exponentials in graphs, we shall compute both A^A and the evaluation map $A^A \times A \xrightarrow{e} A$.

$$\begin{array}{c} V \longrightarrow A^A \\ \hline V \times A \longrightarrow A \\ \hline 2V \longrightarrow A \\ \hline \text{4 vertices} \end{array} \qquad \begin{array}{c} A \longrightarrow A^A \\ \hline A \times A \longrightarrow A \\ \hline A + 2V \longrightarrow A \\ \hline A \longrightarrow A \ , \ 2V \longrightarrow A \\ \hline \text{4 arrows} \end{array}$$

The vertices are defined as follows: let 0 be the vertex of V, s, t those of A and (s,s), (s,t), (t,s), (t,t) be the four morphisms $V \times A \longrightarrow A$ defined in the obvious way. For instance $(t,s)((0,s)) = t$ (first component) and $(t,s)((0,t)) = s$ (second component):



We let $A \times A \xrightarrow{\phi_i} A$ for $i = 1, 2, 3, 4$ be the four arrows. These are defined as follows:



$\phi_i((s,s)) = s,\ \phi_i((t,t)) = t\ \forall i = 1,2,3,4$ and

$$\begin{array}{llll} \phi_1(s,t) = s & \phi_2(s,t) = s & \phi_3(s,t) = t & \phi_4(s,t) = t \\ \phi_1(t,s) = s & \phi_2(t,s) = t & \phi_3(t,s) = s & \phi_4(t,s) = t \end{array}$$

We must compute the incidence relations

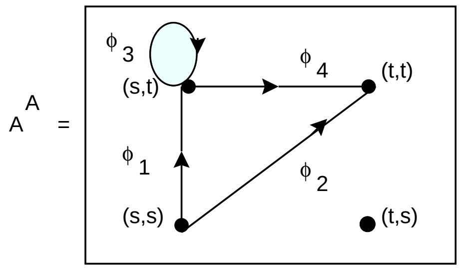
$$\dfrac{\begin{tikzcd} & A \arrow[r, dashed, "\phi"] & A^A \\ V \arrow[ur, "s"] \arrow[urr, dashed, "\phi.s"'] & & \end{tikzcd}}{\begin{tikzcd} & A \times A \arrow[r, "\phi"] & A \\ V \times A \arrow[ur, "\ddot{s}=s\times 1_A"] \arrow[urr, "\phi\circ\ddot{s}"'] & & \end{tikzcd}} \qquad \dfrac{\begin{tikzcd} & A \arrow[r, dashed, "\phi"] & A^A \\ V \arrow[ur, "t"] \arrow[urr, dashed, "\phi.t"'] & & \end{tikzcd}}{\begin{tikzcd} & A \times A \arrow[r, "\phi"] & A \\ V \times A \arrow[ur, "\ddot{t}=t\times 1_A"] \arrow[urr, "\phi\circ\ddot{t}"'] & & \end{tikzcd}}$$

For instance, let us compute the source and the target of ϕ_1 and ϕ_2:

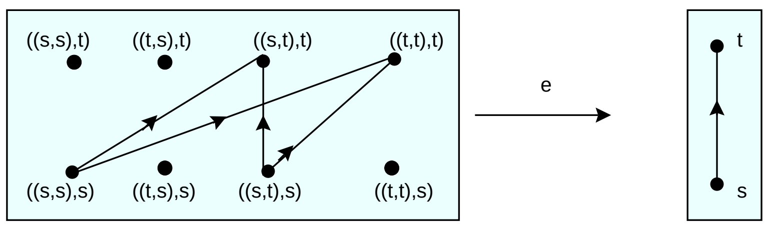
$$\begin{array}{l} \phi_1.s(0,s) = \phi_1\circ\ddot{s}\ (0,s) = \phi_1(s,s) = s \\ \phi_1.s(0,t) = \phi_1\circ\ddot{s}\ (0,t) = \phi_1(s,t) = s \\ \quad \text{hence} \quad \phi_1.s = (s,s) \\ \phi_1.t(0,s) = \phi_1\circ\ddot{t}\ (0,s) = \phi_1(s,s) = s \\ \phi_1.t(0,t) = \phi_1\circ\ddot{t}\ (0,t) = \phi_1(t,t) = t \\ \quad \text{hence} \quad \phi_1.t = (s,t) \end{array}$$

$$\begin{array}{l} \phi_2.s(0,s) = \phi_2\circ\ddot{s}\ (0,s) = \phi_2(s,s) = s \\ \phi_2.s(0,t) = \phi_2\circ\ddot{s}\ (0,t) = \phi_2(s,t) = s \\ \quad \text{hence} \quad \phi_2.s = (s,s) \\ \phi_2.t(0,s) = \phi_2\circ\ddot{t}\ (0,s) = \phi_2(t,s) = t \\ \phi_2.t(0,t) = \phi_2\circ\ddot{t}\ (0,t) = \phi_2(t,t) = t \\ \quad \text{hence} \quad \phi_2.t = (t,t) \end{array}$$

Similarly we compute ϕ_3 and ϕ_4 and obtain the following graph:



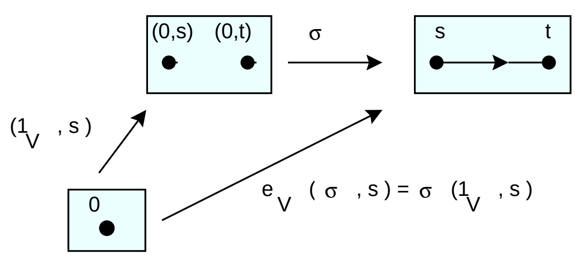
Let us turn to the evaluation: $A^A \times A \xrightarrow{e} A$.



The map e_A is the only map that sends the arrows of the product into the unique arrow of A. (Notice that such a map exists since the product $A^A \times A$ has no loops.) We now compute $e_V(\sigma, s)$ where σ is any of the following morphisms (s,s), (s,t), (t,s), (t,t). By definition $e_V(\sigma, s) = \sigma(1_V, s)$

$$\begin{tikzcd}
 & V \times A \arrow[r, "\sigma"] & A \\
V \arrow[ur, "{(1_V,s)}"] \arrow[urr, "{e_V(\sigma,s)=\sigma\ (1_V,s)}"'] & &
\end{tikzcd}$$

Graphically



For instance, $e_V((t,s),t) = (t,s)(1_V,t) = (t,s)(0,t) = s$. By identifying $(1_V, s)$ and $(1_V, t)$ with s and t, respectively, we conclude that $e_V(\sigma, s) = \sigma(s)$ and $e_V(\sigma, t) = \sigma(t)$. In other words, e_V is the ordinary evaluation.

WINDOW 4.3.1

Infinitesimals in Geometry

In the 60's, Abraham Robinson and others used methods of Model Theory to create Non-standard Analysis, the first rigorous theory of infinitesimals. However, this theory dealt only with the *invertible* infinitesimals used in Analysis and left out of the picture the *nilpotent* infinitesimals which had been used in Geometry by people such as Sophus Lie and Elie Cartan without mentioning physicists and engineers. In 1967 Lawvere, building on previous work by Ehresmann, Weil and Grothendieck, launched a program of creating rigorous foundations for Continuum Mechanics with, as a first step, an intrinsic, coordinate free, limit free theory of differential geometry. Such a theory, known as Synthetic Differential Geometry, is possible in a category with exponentials and explicit infinitesimal structures.

The following simple considerations will show how to construct the tangent bundle of the sphere.

Consider the sphere S^2 as an operation on rings, which associates with any ring R the sphere built on R, i.e.,

$$S^2(R) = \{(a, b, c) \in R^3 | a^2 + b^2 + c^2 = 1\}.$$

In particular, the (ordinary) sphere is the sphere built on $\mathbb{R}$.

Notice that if $f : A \longrightarrow B$ is a ring homomorphism, then we have an obvious map $S^2(f) : S^2(A) \longrightarrow S^2(B)$ which sends (a, b, c) into $(f(a), f(b), f(c))$.

Then the tangent bundle is the sphere S^2, but built on the dual numbers $\mathbb{R}[\epsilon]$ (whose elements are of the form $a + b\epsilon$ with $\epsilon^2 = 0$) and whose structural map is $S^2(\pi)$, where $\pi : \mathbb{R}[\epsilon] \longrightarrow \mathbb{R}$ sends the element $a + b\epsilon$ into a.

In fact $S^2(\mathbb{R}[\epsilon]) = \{(a + u\epsilon, b + v\epsilon, c + w\epsilon) | (a + u\epsilon)^2 + (b + v\epsilon)^2 + (c + w\epsilon)^2 = 1\}$. Carrying out the arithmetical computation the sum of squares is 1 iff $a^2 + b^2 + c^2 = 1$ and $au + bv + cw = 0$. Thus, we may rewrite

$S^2(\mathbb{R}[\epsilon]) = \{(\vec{r},\vec{v}) | \vec{r}.\vec{r} = 1 \text{ and } \vec{r}.\vec{v} = 0\}$ where π becomes the projection into the first component. We recognize the fiber bundle of the sphere.

Of course, S^2 is a presheaf from the dual of the category of rings into *Sets*. Since the category of rings is a proper class, we shall consider as generic figures the dual $\mathbb{C}$ of the category of finitely presented $\mathbb{R}$-rings, i.e., rings of the form

$$\mathbb{R}[X_1, X_2, \ldots, X_n]/(f_1, f_2, \ldots, f_n)$$

Now, the main point is that the tangent bundle may be expressed as an exponential

$$(S^2)^D \xrightarrow{\pi} S^2$$

where D is the presheaf representable by the dual numbers. We can think of $\pi^{-1}(\vec{r})$ as the set of infinitesimal curves passing trough $\vec{r}$. Under the assumption that an infinitesimal curve is a line, we may identify this set with the tangent plane at $\vec{r}$.

This theory has been developed by several authors since 1967. The interested reader may consult the following textbooks [3], [14], [20], [38]. The first three concentrate on the synthetic theory, whereas the last one emphasizes topos-theoretic models including some with both invertible and nilpotent infinitesimals.

`EXERCISE 4.3.1`

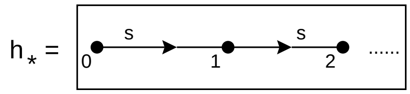
Show that $(S^2)^D(\mathbb{R}) \simeq S^2(\mathbb{R}[\epsilon])$

`EXERCISE 4.3.2`

(1) Calculate and represent graphically the terminal object of $Sets^{\mathbb{C}^{op}}$ ($\mathbb{C}$-sets) where $\mathbb{C}$ is any of the category described in exercise 1.2.2.

(2) Calculate the product of:

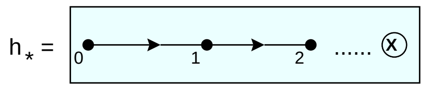
(a) $h_* \times h_*$ where



is the generic chain

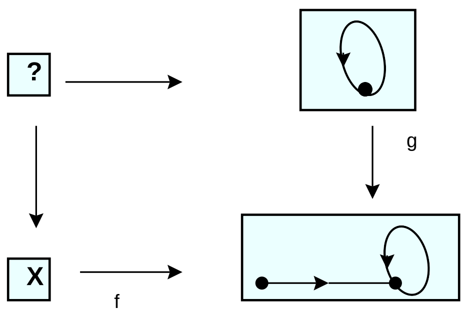
(b) $h_V \times h_V$ and $h_V \times h_A$ where h_V and h_A are the representable functors of the reflexive graphs.

(c) $h_* \times h_*$ where



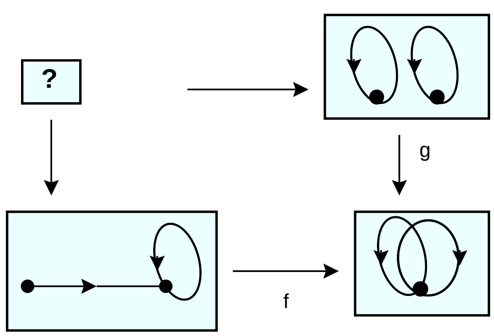
is a possible representation of the generic chain of the monoid described in exercise 1.2.2 (e)

(3) (a) Describe the following pullbacks in *Esets*



where X is any *Esets*

(b) Describe the following pullbacks in *Graphs*



first supposing that f is surjective then that it is not.

(c) A *2-coloring* of a graph X is a couple (R, B) such that $R \cup B =$ vertices of X and $R \cap B = \emptyset$. (We can thing of R as the 'red vertices' and B as the 'blue vertices'). Define a graph G which classifies the 2-coloring of X in the sense that for all X there is a bijection

$$\frac{X \longrightarrow G}{\text{2-coloring of } X}$$

Assume that X has a 2-coloring (R, B). Find $G' \xrightarrow{g} G$ such that the pullback $X \times_G G'$ would be the biggest sub-graph whose vertices are only the red ones; similarly, find $G' \xrightarrow{g} G$ such that the pullback $X \times_G G'$ would be the sub-graph containing only the blue and the red vertices.

5 Generic figures

5.1 What is a generic figure?

The whole book has been built on the notion of generic figure: generic figures for *Sets*, *Bisets*, *Bouquets*, *Graphs*, etc. Although rather intuitive, this notion was left undefined. Two questions arise naturally: when is an arbitrary category $\mathbb{A}$ a category of presheaves? And: how to define the category of generic figures for a category of presheaves?

The answer to the first question is given by a theorem of Roos (see [1, p.415]). Unfortunately, this result goes beyond the level of this book and will not be dealt with.

To make the second question more precise, it seems reasonable to ask the following two properties of the category $\mathbb{G}$ of generic figures for a category of presheaves $Sets^{\mathbb{C}^{op}}$

(i) It should generate the original category in the sense that

$$Sets^{\mathbb{G}^{op}} = Sets^{\mathbb{C}^{op}}$$

(ii) The category $\mathbb{G}$ should be recovered from $Sets^{\mathbb{C}^{op}}$ by means of *intrinsic* properties.

Remark 5.1.1

> Property (ii) allows us to talk about *generic sets*, *generic bouquets*, *generic graphs*, etc.
>
> The property of being representable is not intrinsic. Thus $\mathbb{C}$ cannot be recovered, in general, from $Sets^{\mathbb{C}^{op}}$. Rather, it plays a role similar to that of a basis in topology.

We shall prove that, up to equivalence, there is exactly one category $\mathbb{G}$ satisfying the properties (i) and (ii).

To find $\mathbb{G}$ we shall use the analytic method, as so often done in this work. We look for all the 'intrinsic' properties (i.e., formulated in terms of $Sets^{\mathbb{C}^{op}}$) that the representables have. A generic figure should be one that has all these properties. Up to now we have found only one: representables are connected. However it is clear that connected $\mathbb{C}$-Sets do not satisfy (i). A stronger property is that of being irreducible. A $\mathbb{C}$-set X is *irreducible* (also called absolutely prime or just prime in [35]) iff whenever $X = \bigvee_\alpha X_\alpha$, there is an α such that $X = X_\alpha$. Notice that an irreducible is non-empty: 0 is the $\bigvee$ of the empty family, so there cannot be any $\alpha \in \emptyset$ such that...

Irreducibility is indeed stronger than connectedness:

Proposition 5.1.2 *The irreducible $\mathbb{C}$-sets are connected*

Proof.
Left as exercise.

Proposition 5.1.3 *The representable $\mathbb{C}$-sets are irreducible*

Proof.
Assume that $h_C = \bigvee_\alpha X_\alpha$. Then the generic C-figure 1_C is a C-figure of some X_α. But this implies that $h_C = X_\alpha$.

Once again the irreducible do not satisfy (i) (see the next first exercise)

`EXERCISE 5.1.1`

(1) Show that $\mathbb{1}$ in *Graphs* is irreducible. Show, furthermore, that $Sets^{\mathbb{D}^{op}}$ is not equivalent to *Graphs* where $\mathbb{D}$ is the full subcategory of *Graphs* whose objects are V, A and $\mathbb{1}$.

(2) Show that the frame $Sub(X)$ is generated by the irreducibles. Furthermore, show that Heyting and co-Heyting operations in $Sub(X)$ may be described in terms of irreducibles as follows. If p is an irreducible,

$$p \leq (A \longrightarrow B) \text{ iff } p \leq A \Rightarrow p \leq B$$

$$p \leq (A \setminus B) \text{ iff } p \leq A \wedge p \not\leq B$$

(see [35])

The following characterizes the irreducibles:

Proposition 5.1.4 *The irreducibles are precisely the quotients of representables*

Proof.
Let

$$h_C \overset{q}{\twoheadrightarrow} X$$

be a quotient, i.e., a morphism q such that $\exists_q(h_C) = X$ and let $X = \bigvee_\alpha X_\alpha$. These data gives rise to the following diagram

$$\begin{tikzcd} h_C \arrow[r, two heads, "q"] & X = \bigvee_\alpha X_\alpha \\ q^*(X_\alpha) \arrow[u, hook] & X_\alpha \arrow[u, hook] \end{tikzcd}$$

Then $h_C = \bigvee_\alpha q^*(X_\alpha)$ (indeed, these $\mathbb{C}$-sets have the same F-figures, since $(q^*)_F$ is the ordinary inverse image and preserves unions). By the irreducibility of representables, $h_C = q^*(X_\alpha)$ for some α. Therefore, $X = \exists_q(h_c) = \exists_q q^*(X_\alpha) = X_\alpha$, as can be easily checked.

Assume that X is irreducible. For each $h_C \overset{\sigma}{\longrightarrow} X$, define the sub $\mathbb{C}$-set $< \sigma > \hookrightarrow X$ of X by the formula

$$< \sigma > (C') = \{h'_C \overset{\sigma'}{\longrightarrow} X \mid \exists f : C' \longrightarrow C \ \sigma \circ h_f = \sigma'\}.$$

Clearly, $X = \bigvee\{< \sigma > \mid \sigma : h_C \longrightarrow X\}$. Since X is irreducible, $X =< \sigma >$ for some σ. We claim that $\sigma : h_C \longrightarrow X$ is surjective. It is enough to show this point-wise: let $\sigma' \in X(C')$. By definition of $< \sigma >$, there is some $f : C' \longrightarrow C$ such that $\sigma.f = \sigma'$. This finishes the proof.

A still stronger property is that of continuity: a $\mathbb{C}$-set X is *continuous* iff any map $X \xrightarrow{f} colim_\alpha F_\alpha$ factors through some canonical $F_\alpha \xrightarrow{\eta_\alpha} colim_\alpha F_\alpha$.

Proposition 5.1.5 *A continuous $\mathbb{C}$-set is irreducible*

Proof.
Obvious. Notice that a continuous $\mathbb{C}$-set is automatically non-empty: 0 may be written as the colimit of the empty family.

Proposition 5.1.6 *The representable $\mathbb{C}$-sets are continuous*

Proof.
Let $h_C \xrightarrow{f} colim_\alpha F_\alpha$ be an arbitrary map. By Yoneda and the fact that colimits are computed point-wise, f may be canonically identified with an element of $colim_\alpha(F_\alpha(C))$. The conclusion follows from the way that *colimits* are computed in *Sets* (see section 4.2)

We shall see that this is as far as we need to go: continuity is the strongest 'intrinsic' property that representables have. Does this property characterize representables? The answer is no, but almost...

Proposition 5.1.7 *The continuous $\mathbb{C}$-sets are precisely the retracts of representables*

Proof.
Let X be a retract of h_C. Thus, we have a diagram

$$\begin{tikzcd} X \arrow[r, hook, shift right, "i"'] & h_C \arrow[l, two heads, shift right, "r"'] \end{tikzcd}$$

with $ri = 1_X$. As a notational convention, we use ' $\hookrightarrow$ ' for monomorphism and ' $\twoheadrightarrow$ ' for epimorphism. Let $X \xrightarrow{f} colim_\alpha F_\alpha$ be an arbitrary map. We claim that f factors through some canonical $F_\alpha \xrightarrow{\eta_\alpha} colim_\alpha F_\alpha$. Indeed, the map

$$h_C \xrightarrow{fr} colim_\alpha F_\alpha$$

factors through some $F_\alpha \xrightarrow{\eta_\alpha} colim_\alpha F_\alpha$ as shown in the diagram (since h_C is continuous):

$$\begin{tikzcd}
X \arrow[r, hook, shift right, "i"'] \arrow[ddr, "f"'] & h_C \arrow[l, two heads, shift right, "r"'] \arrow[dd, "fr"] \arrow[dr, "g"] & \\
& & F_\alpha \arrow[dl, "\eta_\alpha"] \\
& colim_\alpha F_\alpha &
\end{tikzcd}$$

Thus, $\eta_\alpha g = fr$. But this implies that f factors through η_α, since

$$\eta_\alpha(gi) = fri = f.$$

To prove the converse, assume that $X \in Sets^{\mathbb{C}^{op}}$ is continuous. Then the map $X \xrightarrow{1_X} X = colim_{C-->X} h_C$ factors through some $h_C \xrightarrow{r} X$ as shown:

$$\begin{tikzcd}
X \arrow[rr, "1_X"] \arrow[dr, dashed, "i"'] & & X \\
& h_C \arrow[ur, "r"'] &
\end{tikzcd}$$

But this diagram presents X as a retract of h_C :

$$\begin{tikzcd}
X \arrow[r, hook, shift right, "i"'] & h_C \arrow[l, two heads, shift right, "r"']
\end{tikzcd}$$

This concludes the proof.

5.2 Continuous $\mathbb{C}$-sets and the Cauchy completion of the category $\mathbb{C}$

Let $Cont(Sets^{\mathbb{C}^{op}})$ be the full subcategory of $Sets^{\mathbb{C}^{op}}$ consisting of the continuous $\mathbb{C}$-sets. In the previous section, we proved that such functors are retracts of representables. In this section we shall prove that this category is equivalent to the category $\overline{\mathbb{C}}$ of idempotents of $\mathbb{C}$, the *Cauchy completion* or the *Karoubi envelope* of $\mathbb{C}$, defined as follows:

An object of $\overline{\mathbb{C}}$ is a couple (C, e) where $C \xrightarrow{e} C$ is an idempotent: $e^2 = e$. A morphism $(C', e') \xrightarrow{f} (C, e)$ of $\overline{\mathbb{C}}$ is a map $C \xrightarrow{f} C'$ such that $fe' = ef = f$. Notice that $(C', e') \xrightarrow{e'} (C', e')$ is the identity.

Define the functor $K : \overline{\mathbb{C}} \longrightarrow Cont(Sets^{\mathbb{C}^{op}})$ by the following stipulation: given a couple (C, e), define $K(C, e)$ to be the sub $\mathbb{C}$-set of h_C determined by the equalizer diagram:

$$K(C,e) \overset{i}{\hookrightarrow} h_C \underset{1_{h_C}}{\overset{h_e}{\rightrightarrows}} h_C$$

If $(C', e') \xrightarrow{f} (C, e)$ is a morphism, we let $K(f)$ to be the obvious map defined by the universal property of equalizers (using that $h_e \circ h_f = h_f \circ h_{e'} = h_f$) :

$$\begin{tikzcd}
(C,e) & K(C,e) \arrow[r, hook, "i"] & h_C \arrow[r, shift left, "h_e"] \arrow[r, shift right, "1_{h_C}"'] & h_C \\
(C',e') \arrow[u, "f"] & K(C',e') \arrow[u, dashed, "K(f)"'] \arrow[r, hook, "i'"] & h_{C'} \arrow[u, "h_f"] \arrow[r, shift left, "h_{e'}"] \arrow[r, shift right, "1_{h_{C'}}"'] & h_{C'} \arrow[u, "h_f"']
\end{tikzcd} \qquad (*)$$

Proposition 5.2.1 *The functor $K : \overline{\mathbb{C}} \longrightarrow Cont(Sets^{\mathbb{C}^{op}})$ is an equivalence of categories*

We shall prove that K is faithful, full and essentially surjective. The key to prove all of these properties is the following

Lemma 5.2.2 *Assume that $\mathbb{A}$ is a category with equalizers. If $A \xrightarrow{e} A$ is an idempotent, then there is a diagram*

$$E \xrightarrow{i} A \underset{1_A}{\overset{e}{\rightrightarrows}} A \xrightarrow{r} E$$

such that

(i) $ri = 1_E,\ ir = e$

(ii) The diagrams

$$E \xrightarrow{i} A \underset{1_A}{\overset{e}{\rightrightarrows}} A \qquad A \underset{1_A}{\overset{e}{\rightrightarrows}} A \xrightarrow{r} E$$

are an equalizer and a coequalizer, respectively.

Proof.
Left as an exercise.

Proof (of proposition 5.2.1).
• K is faithful: if $f,\ g : C' \longrightarrow C$, then $iK(f) = h_f i'$ and $iK(g) = h_g i'$, by definition of K. (See the preceding diagram (*).) Assume that $K(f) = K(g)$. Then $h_f i' = h_g i'$ and since i' is monic, $h_f = h_g$ and hence, $f = g$.
• K is full: assume that $K(C', e') \xrightarrow{\Phi} K(C, e)$ is given. Define $C' \xrightarrow{f} C$ by the formula $h_f = i\Phi r'$. Clearly $h_e \circ h_f = h_f \circ h_{e'} = h_f$. To show that $K(f) = \Phi$, it is enough to check that $i\Phi = h_f i'$. But by the lemma 5.2.2, it is clear that: $h_f i' = i\Phi r' i' = i\Phi$.
•K is essentially surjective (i.e., every object of the codomain is isomorphic to some that comes from the domain via K): assume that

$$X \underset{i}{\overset{r}{\rightleftarrows}} h_C$$

is given with $ri = 1_X$. Define $e = ir$. Since e is an idempotent,

$$X \overset{i}{\hookrightarrow} h_C \underset{1_{h_C}}{\overset{h_e}{\rightrightarrows}} h_C$$

is an equalizer (by the lemma 5.2.2 with $\mathbb{A} = Sets^{\mathbb{C}^{op}}$). But this implies that $X = K(C, e)$ by the very definition of K.

This proposition shows that $\overline{\mathbb{C}}$ can be recovered intrinsically from $Sets^{\mathbb{C}^{op}}$ as being (equivalent to) the full subcategory of $Sets^{\mathbb{C}^{op}}$ consisting of the continuous $\mathbb{C}$-sets.

Notice that we have an obvious functor $I : \mathbb{C} \longrightarrow \overline{\mathbb{C}}$ which sends C into $(C, 1_C)$.

Proposition 5.2.3 *The functor $I : \mathbb{C} \longrightarrow \overline{\mathbb{C}}$ induces by composition an equivalence of categories*

$$I^* : Sets^{\overline{\mathbb{C}}^{op}} \xrightarrow{\sim} Sets^{\mathbb{C}^{op}}$$

Proof.
We shall prove that I^* is faithful, full and essentially surjective.
• I^* is faithful: Let $\Phi,\ \Psi : X \longrightarrow Y$ be natural transformations. Define $\phi = \Phi \circ I$ and $\psi = \Psi \circ I$. Assume that we are given $(C, e) \in \overline{\mathbb{C}}$. The diagram

$$\begin{tikzcd} (C,e) \arrow[r, hook, shift right, "i"'] & (C,1_C) \arrow[l, two heads, shift right, "r"'] \end{tikzcd}$$

(where we use 'i' and 'r' for the same map e to distinguish domains and codomains) gives rise to a diagram

$$\begin{tikzcd} X(C,e) \arrow[r, hook, shift right, "R_X"'] \arrow[d, "\Phi_{(C,e)}"'] & X(C) \arrow[l, two heads, shift right, "I_X"'] \arrow[d, "\phi_C"] \\ Y(C,e) \arrow[r, hook, shift right, "R_Y"'] & Y(C) \arrow[l, two heads, shift right, "I_Y"'] \end{tikzcd}$$

where $I_X = X(i)$, $R_X = X(r)$, $I_Y = Y(i)$ and $R_Y = Y(r)$ are such that $I_Y \circ \phi_C = \Phi_{(C,e)} \circ I_X$ and $I_Y \circ \psi_C = \Psi_{(C,e)} \circ I_X$. Assume that $\phi = \psi$. Then $\Phi_{(C,e)} \circ I_X = \Psi_{(C,e)} \circ I_X$. Since I_X is epi, $\Phi_{(C,e)} = \Psi_{(C,e)}$. Thus, $\Phi = \Psi$.

- I^* is full: let $X \in Sets^{\overline{\mathbb{C}}^{op}}$ and let $X \circ I \xrightarrow{\phi} Y \circ I$ be a natural transformation. We shall define $X \xrightarrow{\Phi} Y$ such that $\Phi \circ I = \phi$. Assume (C,e) given. Then we have the following diagram

$$\begin{tikzcd} (C,e) \arrow[r, "i"] & (C,1_C) \arrow[r, shift left, "e"] \arrow[r, shift right, "1_C"'] & (C,1_C) \arrow[r, "r"] & (C,1_C) \end{tikzcd}$$

Let I_X, I_Y, R_X and R_Y as before. Define

$$\Phi_{(C,e)} = I_Y \circ \phi_C \circ R_X$$

We show that $\left(\Phi_{(C,e)}\right)_{(C,e)}$ is a natural transformation by first proving the following

Lemma 5.2.4 *Consider the diagram*

$$\begin{tikzcd} A \arrow[rrr, "f"] \arrow[ddd, "u"'] & & & B \arrow[dl, "k"] \arrow[ddd, "v"] \\ & A' \arrow[ul, "i"] \arrow[r, "f'"] \arrow[d, "u'"'] & B' \arrow[d, "v'"] & \\ & C' \arrow[r, "g'"'] \arrow[dl, "j"'] & D' & \\ C \arrow[rrr, "g"'] & & & D \arrow[ul, "l"'] \end{tikzcd}$$

Assume that

$$\begin{array}{ll} vf = gu & (1) \\ v'k = lv & (2) \\ kfi = f'\ , \quad lgj = g' & (3) \\ ui = ju' & (4) \end{array}$$

then $g'u' = v'f'$

Proof.
Simple computation: $v'f' = v'kfi = lvfi = lgui = lgju' = g'u'$.

We return to the proof of proposition 5.2.3 to show that $\left(\Phi_{(C,e)}\right)_{(C,e)}$ is a natural transformation. Let $(C',e') \xrightarrow{f} (C,e)$ be a morphism of $\overline{\mathbb{C}}$. We must prove that the inner square of the diagram

$$\begin{tikzcd}
X \circ I(C) \arrow[rrr, "\phi_C"] \arrow[ddd, "X \circ I(f)"'] & & & Y \circ I(C) \arrow[ddd, "Y \circ I(f)"] \arrow[dl, "I_Y"'] \\
& X(C,e) \arrow[ul, "R_X"'] \arrow[r, "\Phi_{(C,e)}"] \arrow[d, "X(f)"'] & Y(C,e) \arrow[d, "Y(f)"] & \\
& X(C',e') \arrow[dl, "R'_X"'] \arrow[r, "\Phi_{(C',e')}"'] & Y(C',e') & \\
X \circ I(C') \arrow[rrr, "\phi_{C'}"'] & & & Y \circ I(C') \arrow[ul, "I'_Y"']
\end{tikzcd}$$

is commutative. But this follows from the lemma. In fact, (1) holds because ϕ is a natural transformation, (2) and (4) because X and Y are natural transformations, while (3) holds by definition of Φ. The fact that $\Phi \circ I = \phi$ follows by definition of Φ.

• I^* is essentially surjective: assume that $X : \mathbb{C}^{op} \longrightarrow Sets$. We define

$$\overline{X} : \overline{\mathbb{C}}^{op} \longrightarrow Sets$$

on an object (C,e) of $\overline{\mathbb{C}}$ by requiring that

$$\begin{tikzcd}
\overline{X}(C,e) \arrow[r, hook] & X(C) \arrow[r, shift left, "X(e)"] \arrow[r, shift right, "1_{X(C)}"'] & X(C)
\end{tikzcd}$$

is an equalizer. On a morphism $(C', e') \xrightarrow{f} (C, e)$, $\overline{X}(f)$ is defined by the universal property of equalizers:

$$
\begin{tikzcd}
(C,e) & \overline{X}(C,e) \arrow[r, hook] \arrow[d, dashed, "\overline{X}(f)"'] & X(C) \arrow[r, shift left, "X(e)"] \arrow[r, shift right, "1_{X(C)}"'] \arrow[d, "X(f)"'] & X(C) \arrow[d, "X(f)"] \\
(C',e') \arrow[u, "f"] & \overline{X}(C',e') \arrow[r, hook] & X(C') \arrow[r, shift left, "X(e')"] \arrow[r, shift right, "1_{h_{X(C')}}"'] & X(C')
\end{tikzcd}
$$

It is clear that $\overline{X} \circ I = X$. Further details are left as exercise.□

The last two propositions show that continuity is the strongest intrinsic property that representables have. Thus it is reasonable to identify the category of generic figures of $Sets^{\mathbb{C}^{op}}$ with the Cauchy completion $\overline{\mathbb{C}}$ of $\mathbb{C}$. Returning to our examples, this means that with the exception of the reflexive graphs, the generic figures are precisely the representables (since in all these cases the representables are their own Cauchy completion: there are no non-trivial idempotents). On the other hand, the Cauchy completion of

$$
\begin{tikzcd}
V \arrow[r, shift left=2, "s"] \arrow[r, "t"] & A \arrow[l, shift left=2, "l"] \arrow[loop above, "\sigma"] \arrow[loop below, "\tau"]
\end{tikzcd}
$$

is

$$
\begin{tikzcd}
 & & A_1 \arrow[dl, shift right, "\tau"'] \arrow[dl, shift left, "\sigma"] \arrow[dd, shift left, "\tau"] \\
V \arrow[r, shift left=2, "s"] \arrow[r, "t"] & A \arrow[l, shift left=2, "l"] \arrow[loop above, "\sigma"] \arrow[loop below, "\tau"] \arrow[ur, bend right=20, "\sigma"'] \arrow[dr, bend left=20, "\tau"] & \\
 & & A_2 \arrow[ul, shift right, "\tau"] \arrow[ul, shift left, "\sigma"'] \arrow[uu, shift left, "\sigma"]
\end{tikzcd}
$$

Notice, however, that in this case the Cauchy completion is equivalent to the original category. Indeed, the inclusion I of the original category in its Cauchy completion is obviously full and faithful. Furthermore, it is essentially surjective since both A_1 and A_2 are isomorphic to $I(V)$.

Remark 5.2.5 The passage from a category $\mathbb{C}$ to its Cauchy completion $\overline{\mathbb{C}}$ may be described as the process of 'splitting idempotents'. In fact, for every idempotent $C \xrightarrow{e} C$ of the original category we have a new object (C, e) in its Cauchy completion together with maps

$$\begin{tikzcd}(C,e) \arrow[r, hook, shift right, "i"'] & (C,1_C) \arrow[l, two heads, shift right, "r"']\end{tikzcd}$$

such that $ir = e$. (We have used 'i' and 'r' for the same map e to distinguish domains and codomains). Furthermore this splitting is universal in a sense that the interested reader may easily formulate and prove.

EXERCISE 5.2.1

(1) Give graphical representations of the representables of the Cauchy completion of

$$\begin{tikzcd}V \arrow[r, shift left=2, "s"] \arrow[r, "t"] & A \arrow[l, shift left=2, "l"] \arrow[loop above, "\sigma"] \arrow[loop below, "\tau"]\end{tikzcd}$$

(2) Prove that if

$$\begin{tikzcd}E \arrow[r, hook, shift right, "i"'] & A \arrow[l, two heads, shift right, "r"']\end{tikzcd}$$

is a retract (i.e.,$ri = 1_E$) in a category $\mathbb{A}$ the morphism $A \xrightarrow{e} A$ defined by $e = ir$ is an idempotent. Furthermore show that
(i)

$$\begin{tikzcd}E \arrow[r, hook, "i"] & A \arrow[r, shift left, "e"] \arrow[r, shift right, "1_A"'] & A\end{tikzcd}$$

is an equalizer
(ii)

$$\begin{tikzcd}A \arrow[r, shift left, "e"] \arrow[r, shift right, "1_A"'] & A \arrow[r, "r"] & E\end{tikzcd}$$

is a coequalizer

(3) Prove that in any category $\mathbb{A}$, if e is an idempotent on A, the diagram

$$A \underset{1_A}{\overset{e}{\rightrightarrows}} A$$

has an equalizer E if and only if it has a coequalizer Q. Furthermore E and Q are canonically isomorphic.

(4) Prove that if $\mathbb{A}$ has either equalizers or coequalizers, every idempotent is given by a retract.

(5) Show that $Sets^{\mathbb{C}^{op}} \simeq Sets^{\mathbb{D}^{op}}$ iff $\overline{\mathbb{C}} \simeq \overline{\mathbb{D}}$

6 The object Ω of truth values

Let us recall that in sets there is a bijection between the funtions $X \xrightarrow{\phi} \Omega$ and the subsets $Y \subseteq X$, where $\Omega = \{\top, \bot\}$ ('$\top$' stands for true and '$\bot$' stands for false):

$$\frac{X \xrightarrow{\phi} \Omega}{Y \subseteq X}$$

If ϕ is given, $Y = \phi^{-1}(\{\top\})$. We will use the notation $\phi^*(\top)$ for $\phi^{-1}(\{\top\})$. On the other hand if Y is given, $\phi = \chi_Y$, the characteristic function of Y defined as follows:

$$\chi_Y(x) = \begin{cases} \top & \text{if } x \in Y \\ \bot & \text{if } x \notin Y \end{cases}$$

We can express the bijection by saying that 'the subsets of X are classified by the functions of $X \longrightarrow \Omega$' or more simply 'Ω classifies the subsets of a set'.

We will show that for every $\mathbb{C}$ the category of $\mathbb{C}$-sets or, equivalently, the category of functors $Set^{\mathbb{C}^{op}}$ has an object Ω with the 'same' property.

In other words, we shall prove the existence of an object Ω which classifies the sub $\mathbb{C}$-sets of a $\mathbb{C}$ set X in the sense that there is a bijection

$$\frac{X \xrightarrow{\phi} \Omega}{Y \hookrightarrow X}$$

Starting from this property we might compute the F-figures of Ω as follows:

$$\begin{array}{c} F \dashrightarrow \Omega \\ \hline h_F \longrightarrow \Omega \\ \hline X \hookrightarrow h_F \end{array} \quad \begin{array}{l} \text{byYoneda} \\ \text{by the property of } \Omega \end{array}$$

That is to say an F-figure of Ω may be identified with a sub-$\mathbb{C}$-set of the representable h_F. We must define the action of a morphism on the generic figures:

$$\begin{tikzcd} & F \arrow[r, dashed, "\sigma"] & \Omega \\ F' \arrow[ur, "f"] \arrow[urr, dashed, "\sigma.f=?"'] & & \end{tikzcd}$$

Going back to the definition of the figures of Ω, we need to define $\sigma.f$ in the diagram

$$\begin{tikzcd} \sigma \arrow[r, hook] & h_F \\ \sigma.f \arrow[r, hook] & h_{F'} \arrow[u, "h_f"'] \end{tikzcd}$$

We define $\sigma.f = h_f^*(\sigma)$, i.e., the *inverse image* of h_F applied to σ. We recall that

$$(h_f^*(\sigma))(F'') = (h_f)^{-1}_{F''}(\sigma(F'')).$$

Notice that the right hand side refers to the ordinary inverse image of the set $\sigma(F'')$ by the (set-theoretical) function $(h_f)_{F''}$. We have to check that thus defined $\sigma.f$ is a sub-$\mathbb{C}$-set of $h_{F'}$.

Following the analogy with sets, we must obtain a generalization of $\phi^*(\top)$. For every F, let

$$F \overset{\top_F}{\dashrightarrow} \Omega$$

be the largest sub $\mathbb{C}$-set of h_F, namely h_F. Notice that the family $(\top_F)_F$ has the property that $\top_F.f = \top_{F'}$ for a change of figure $F' \xrightarrow{f} F$. In diagrams

$$\begin{tikzcd} & F \arrow[r, dashed, "\top_F"] & \Omega \\ F' \arrow[ur, "f"] \arrow[urr, dashed, "\top_F.f=\top_{F'}"'] & & \end{tikzcd} \qquad (*)$$

We define $\phi^*(\top)$ as the family $(\phi^*_F(\top_F))_F$ where

$$\phi^*_F(\top_F) = \{\sigma \in X(F) : \phi_F(\sigma) = \top_F\}.$$

By using (*), we obtain that $\phi^*(\top)$ is a sub-$\mathbb{C}$-set of X: let $\sigma \in \phi^*_F(\top_F)$, i.e., $\phi_F(\sigma) = \top_F$. Hence

$$\phi_{F'}(\sigma.f) = \phi_F(\sigma).f = \top_F.f = \top_{F'}$$

by (*). Thus $\sigma.f \in \phi^*_{F'}(\top_{F'})$.

Thus the property that Ω classifies sub $\mathbb{C}$-sets of an X-set may be made precise by requiring that the function that sends ϕ into $\phi^*(\top)$) is a bijection:

$$\frac{X \xrightarrow{\phi} \Omega}{Y \subseteq X} \qquad Y = \phi^*(\top)$$

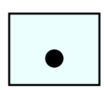
In the next section we shall compute Ω in our examples.

6.1 Computation of the object Ω in the examples

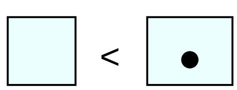
• *Sets.* Let us compute the generic figures of Ω

$$\frac{P \overset{\sigma}{\dashrightarrow} \Omega}{\sigma \hookrightarrow h_P}$$

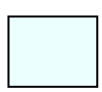
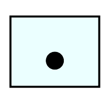
The P-figures are hence the sub-sets of

$h_P =$ 

There are two with the canonical order (inclusion):



Letting

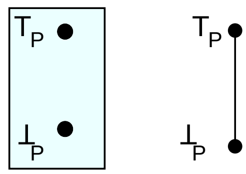
$\perp_P =$  $\top_P =$ 

we have

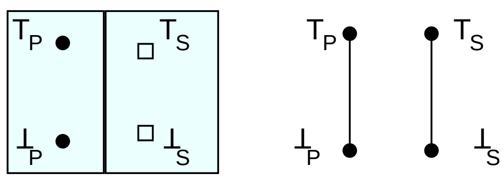
$$\frac{P \dashrightarrow \Omega}{\perp_P, \top_P}$$

with $\perp_P < \top_P$. There are no non trivial changes of figures.

We represent Ω as a set and as a Hasse diagram to indicate the order.

$\Omega =$ 

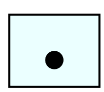
• *Bisets*. Proceeding as in the previous case, we may represent Ω also as a biset and as a Hasse diagram.

$\Omega =$ 

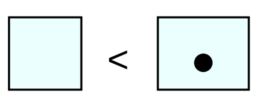
• *Bouquets*. Let us compute the generic figures of Ω

$$\frac{V \overset{\sigma}{\dashrightarrow} \Omega}{\sigma \hookrightarrow h_V}$$

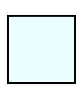
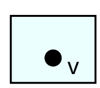
Thus the V-figures are the sub-bouquets of

$h_V =$ 

There are two with the canonical order (inclusion):



Letting

$\perp_V =$  $\top_V =$ 

we have

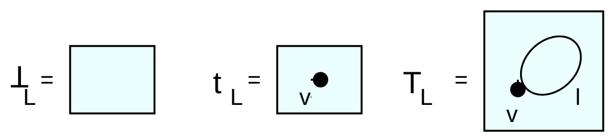
$$\frac{V \dashrightarrow \Omega}{\perp_V, \top_V}$$

with $\perp_V < \top_V$.

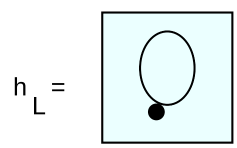
Similarly

$$\frac{L \dashrightarrow \Omega}{\perp_L, t_L, \top_L}$$

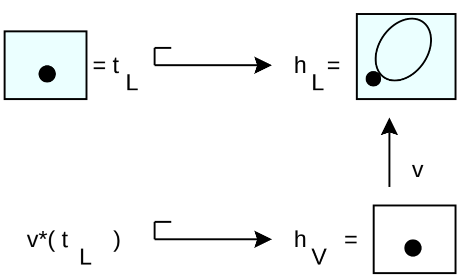
where



are the sub-bouquets of



Notice that $\perp_L < t_L < \top_L$ (under inclusion). We have now to compute the incidence relations:

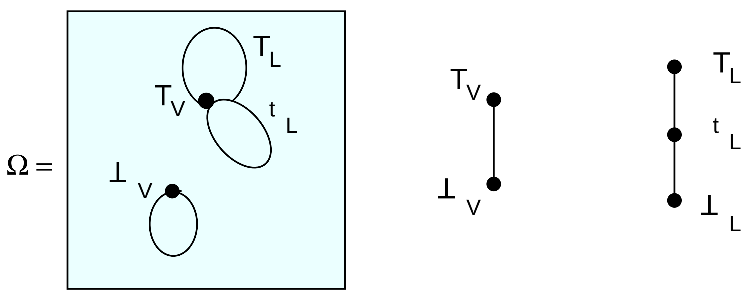


From the definition of $v^*(t_L)$ it follows that its vertices are the vertices of h_V sent by v into the vertices of t_L. But h_V has precisely one vertex and this is sent by v into the only vertex of t_L. Thus $v^*(t_L) = h_V = \top_V$, i.e., $t_L.v = \top_V$.

On the other hand, $v^*(\perp_L)$ does not have any vertices just as $\perp_L$ (and the only vertex of h_V cannot be sent into the empty set). Thus, $\perp_L.v = \perp_V$.

In a similar way we can check that $\top_L.v = \top_V$.

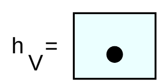
We can summarize all this information by representing Ω graphically as a bouquet and as a Hasse diagram to indicate the (partial) order relation between its elements:



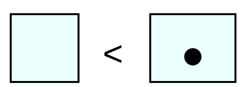
• *Graphs*. Let us compute the generic figures of Ω

$$\frac{V \overset{\sigma}{\dashrightarrow} \Omega}{\sigma \hookrightarrow h_V}$$

The V-figures are hence the sub-graphs of



There are two with the canonical order (inclusion):



Letting

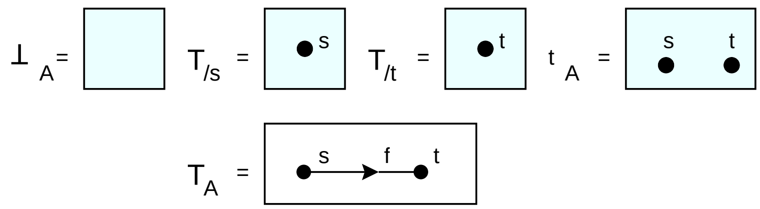
we have

$$\frac{V \dashrightarrow \Omega}{\perp_V, \top_V}$$

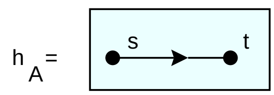
with $\perp_V < \top_V$. Similarly

$$\frac{A \dashrightarrow \Omega}{\perp_A, t_A, \top/s, \top/t, \top_A}$$

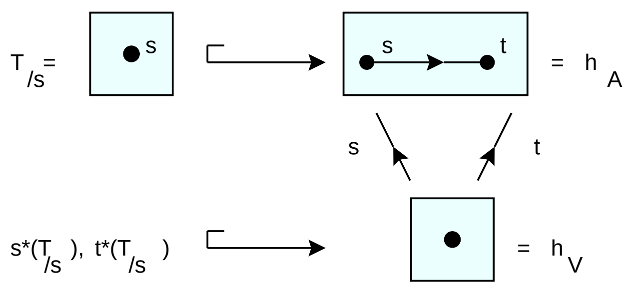
where



are the sub-graphs of



The order relation is given by the inclusion (see the Hasse diagram in the graphical representation of Ω). We have now to compute the incidence relations. Since $\perp_A$ is the empty graph it follows by the argument given for the bouquets that $\perp_A.s = \perp_A.t = \perp_V$ and $\top_A.s = \top_A.t = t_A.s = t_A.t = \top_V$. We will compute $(\top/s).s$ and $(\top/s).t$.

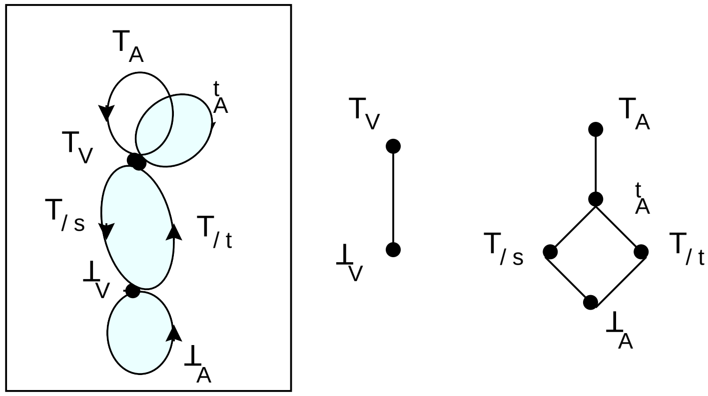


From the definition of $s^*(\top/s)$, it follows that its vertices are the vertices of h_V sent by s into those of $\top/s$. But h_V has exactly one vertex and this is sent by s into the only vertex of $\top/s$. Thus $s^*(\top/s) = h_V$. In other words, $(\top/s).s = \top_V$. On the other hand, $t^*(\top/s) = \perp_V$, since t sends the only vertex of h_V into $\top/t$, not into $\top/s$, i.e., $(\top/s).t = \perp_V$.

Similarly, $(\top/t).t = \top_V$ and $(\top/t).s = \perp_V$.

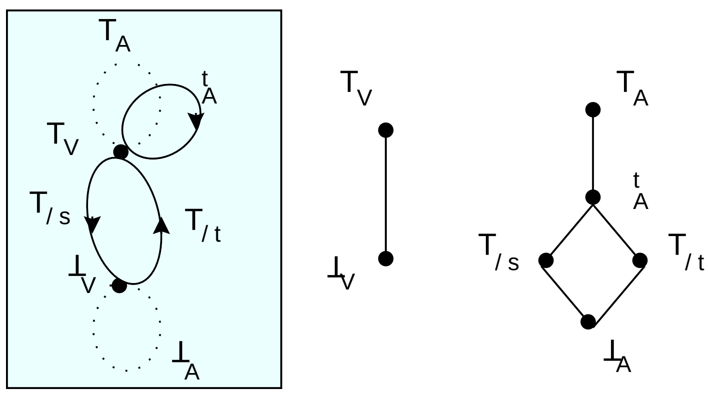
We shall summarize all this information by representing Ω graphically as a graph and as a Hasse diagram to indicate the (partial) order relation between its elements:

Ω =



• *Rgraphs*. Proceeding as in the previous case, we may represent Ω also as a reflexive graph and as a Hasse diagram. The representation is perhaps unexpected:

Ω =

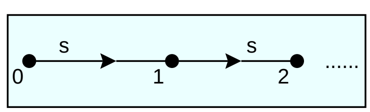


• *Esets*. Let us compute the generic figures of Ω

$$\frac{* \overset{\tau}{\dashrightarrow} \Omega}{\tau \hookrightarrow h_*}$$

The $*$-figures are hence the sub-evolutive sets of

h_* =



where s is the successor function. There are infinitely many, each one being a tail (except for the empty set) of the form $[n,\infty)$, with the canonical order (inclusion):

$$\emptyset \subset .... \subset [2,\infty) \subset [1,\infty) \subset [0,\infty) = h_*$$

We have now to compute the action:

$$\begin{tikzcd} & * \arrow[r, dashed, "{[n,\infty)}"] & \Omega \\ * \arrow[ur, "\sigma"] \arrow[urr, dashed, "{[n,\infty).\sigma}"'] & & \end{tikzcd}$$

In other words, we have to compute $s^*([n,\infty))$ in the following diagram:

$$\begin{tikzcd} {[n,\infty)} \arrow[r, hook] & {[0,\infty) = h_*} \\ s^*([n,\infty)) \arrow[r, hook] & {[0,\infty) = h_*} \arrow[u, "s"'] \end{tikzcd}$$

But since there is only one generic figure, the inverse image in question is the ordinary (or set-theoretical) inverse image:

$$s^*([n,\infty)) = \{x : s(x) \in [n,\infty)\} = [n-1,\infty)$$

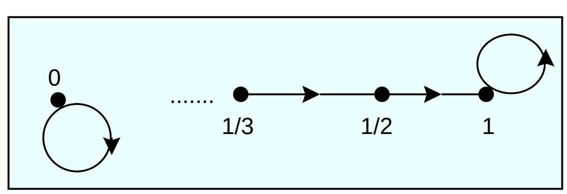
Thus, $[n,\infty).\sigma = [n-1,\infty)$. Let us use the notation '1/n' to denote $[n+1,\infty)$ and '0' to denote $\emptyset$. With this notation, the sought action becomes:

$0.\sigma = 0$

$1/(n+1).\sigma = 1/n$

$1.\sigma = 1$

We shall summarize all this information by representing Ω graphically as an evolutive set with the order induced by the rationals:

$\Omega =$ 

EXERCISE 6.1.1

> Calculate and represent graphically the object Ω of $Sets^{\mathbb{C}^{op}}$ where $\mathbb{C}$ is described in (c), (e), (f), (g) and (h) of exercise 1.2.2.

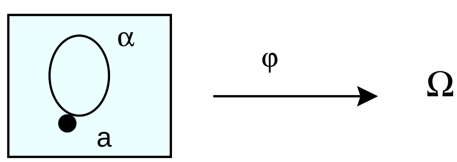
6.2 Ω as a classifier

Let us notice that we have not yet proved that Ω is indeed a classifier of sub $\mathbb{C}$-sets in the sense that there is a bijection.

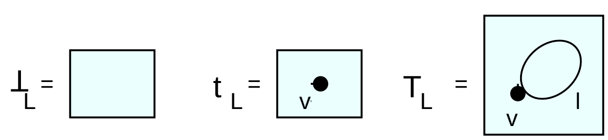
$$\frac{X \xrightarrow{\phi} \Omega}{Y \subseteq X} \qquad Y = \phi^*(\top)$$

What this means, essentially, is that a morphism of $\mathbb{C}$-sets is completely determined once we know what F-figures are sent into $\top_F$ for every F. Let us consider two examples, the first in the bouquets and the other in the graphs:

- Consider the morphism of bouquets



Then $\phi(\alpha)$ is one among



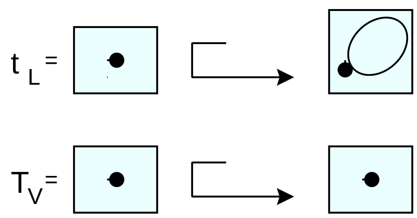
and $\phi(a)$ is one among

Clearly:

> the loop l is in $\phi(\alpha)$ iff $\phi(\alpha) = \phi(\alpha.1_L) = \top_L$

the vertex v is in $\phi(a)$ iff $\phi(a) = \phi(\alpha.v) = \top_V$.

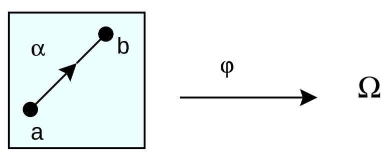
Remark 6.2.1 Recall that a sub $\mathbb{C}$-set Y of X is really an inclusion $Y \hookrightarrow X$, although we usually write only the domain. Thus



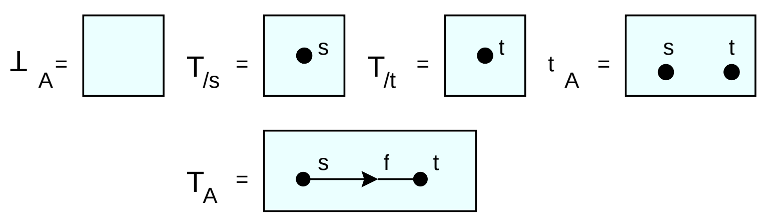
are different inclusions with the same domain.

Assume that we know that only a is sent into $\top_V$, then, $\phi(\alpha)$ has the vertex, but not the loop, i.e., $\phi(\alpha) = t_L$.

- Consider the morphism of graphs



Then $\phi(\alpha)$ is one among



and $\phi(a)$ is one among

Clearly:

the arrow f is in $\phi(\alpha)$ iff $\phi(\alpha) = \phi(\alpha.1_A) = \top_A$

the source s is in $\phi(\alpha)$ iff $\phi(a) = \phi(\alpha.s) = \top_V$

the target t is in $\phi(\alpha)$ iff $\phi(b) = \phi(\alpha.t) = \top_V$

Assume that b is the only element sent into $\top_V$. Then $\phi(\alpha)$ has the target and nothing else, i.e., $\phi(\alpha) = \top/t$. Obviously $\phi(a) = \perp_V$.

In these examples, at least, ϕ is determined by $Y = \phi^*(\top)$, that is to say by the F-figures that are sent into $\top_F$ for each F. In fact if σ is an F-figure of X and $F' \xrightarrow{f} F$ is a change of figure,

$$\begin{aligned} f \in \phi(\sigma)(F') &\Leftrightarrow \phi(\sigma.f) = \top_{F'} \\ &\Leftrightarrow \sigma.f \in (\phi^*(\top))_{F'} \\ &\Leftrightarrow \sigma.f \in Y(F') \end{aligned}$$

This motivates the definition of the *characteristic function* of Y in the proof of the following

Theorem 6.2.2 *Let $\mathbb{C}$ be a category and let X be a $\mathbb{C}$-set. Then there is a bijection*

$$\frac{X \xrightarrow{\phi} \Omega}{Y \hookrightarrow X} \qquad Y = \phi^*(\top)$$

Proof.
We shall define the *characteristic function* of Y:

$$\chi_Y : X \longrightarrow \Omega$$

which generalizes the corresponding one for sets. To do this, we have to define for every F a set-theoretical function $(\chi_Y)_F : X(F) \longrightarrow \Omega(F)$. Let $\sigma \in X(F)$. Then $(\chi_Y)_F(\sigma)$ should be an F-figure of Ω, i.e., a sub $\mathbb{C}$-set of the representable h_F, by definition of Ω. Let $f : F' \longrightarrow F$ be an F'-figure of h_F. Then we define

$$f \in (\chi_Y)_F(\sigma)(F') \text{ iff } \sigma.f \in Y(F')$$

From the axioms for an action, it follows that χ_Y is indeed a morphism of $\mathbb{C}$-sets: $(\chi_Y)_{F'}(\sigma.f) = (\chi_Y)_F(\sigma).f$

We claim that these operations are inverse of each other: $\chi_{\phi^*(\top)} = \phi$ and $(\chi_Y)^*(\top) = Y$. The proof will use the following simple fact stating that 1_F is a *generic* element and will be used on other occasions, without mention:

Lemma 6.2.3 *Let $X \hookrightarrow h_F$ be a sub $\mathbb{C}$-set of the representable h_F. Then $X = h_F$ iff $1_F \in X(F)$.*

Let us return to the proof of the theorem. We first show that $\chi_{\phi^*(\top)} = \phi$.

$$
\begin{aligned}
f \in (\chi_{\phi^*(\top)})_F(\sigma)(F') &\iff \sigma.f \in \phi^*(\top)(F') \\
&\iff \phi(\sigma.f) = \top_{F'} \\
&\iff \phi(\sigma).f = \top_{F'} \\
&\iff \phi(\sigma).f = h_{F'} \\
&\iff 1_{F'} \in (\phi(\sigma).f)(F') \\
&\iff 1_{F'} \in {h_f}^*(\phi(\sigma)(F')) \\
&\iff h_f(1_{F'}) \in \phi(\sigma)(F') \\
&\iff f \in \phi(\sigma)(F')
\end{aligned}
$$

Let us turn to the proof that $(\chi_Y)^*(\top) = Y$.

Let σ be an F-figure of X. Then

$$
\begin{aligned}
\sigma \in (\chi_Y)^*(\top)(F) &\iff \sigma \in ((\chi_Y)_F)^*(\top_F) \\
&\iff (\chi_Y)_F(\sigma) = \top_F \\
&\iff (\chi_Y)_F(\sigma) = h_F \\
&\iff 1_F \in (\chi_Y)_F(\sigma)(F) \\
&\iff \sigma.1_F \in Y(F) \\
&\iff \sigma \in Y(F)
\end{aligned}
$$

EXERCISE 6.2.1

1. Show that the family $(\top_F)_F$ may be viewed as a morphism

$$\top : \mathbb{1} \longrightarrow \Omega$$

of $\mathbb{C}$-sets from $\mathbb{1}$ into Ω.

2. Let $\phi, \psi : X \longrightarrow \Omega$ be $\mathbb{C}$-morphisms. Show that the following are equivalent

   (i) $\forall F \in \mathbb{C}\ \phi_F \leq \psi_F$

(ii) $\forall F \in \mathbb{C}\ \forall \sigma \in X(F)\ [\phi_F(\sigma) = \top_F \Rightarrow \psi_F(\sigma) = \top_F]$

WINDOW 6.2.1

Grothendieck toposes versus elementary toposes

Gathering what we have shown up to this point we may say that whenever $\mathbb{C}$ is a small category, the category of $\mathbb{C}$-sets, $Sets^{\mathbb{C}^{op}}$, has the following properties:

(1) it has finite limits, i.e., terminal object $\mathbb{1}$ and pullbacks

(2) it has exponentials, i.e., the functor $(\ ) \times X$ has a right adjoint

(3) it has an object Ω which classifies subobjects

A category with these properties is called an *elementary topos*.

It is remarkable the power of this finite list of axioms. One can prove that an elementary topos has finite colimits. Furthermore, the Heyting operations: $\vee$, $\wedge$, $\rightarrow, \neg$, $\top$ and $\perp$ are definable as operations on powers of Ω, the last two as maps $\mathbb{1} \longrightarrow \Omega$. Indeed, an elementary topos has the same expressive power as intuitionistic type theory.

The foremost example of elementary toposes are Grothendieck toposes. These are subcategories of $\mathbb{C}$-sets defined in terms of a Grothendieck topology. A *Grothendieck topology* is a small category $\mathbb{C}$ together with a notion of formal cover given as a set of families $Cov(C)$ of maps with C as codomain, for each $C \in \mathbb{C}$. These maps are supposed to satisfy some conditions of the type 'isomorphisms cover', 'covers are stable under pullbacks'. We define a presheaf to be a *sheaf* if it *believes* that the stipulated formal cover are covers. Needless to say, this is a precise mathematical notion whose gist is the description given informally. We let $Sh(\mathbb{C}, Cov)$ be the full subcategory of the category of presheaves consisting of sheaves. One can show that $Sh(\mathbb{C}, Cov)$ is a reflexive subcategory of $Sets^{\mathbb{C}^{op}}$ in the sense that the inclusion functor has an exact left adjoint ('the associated sheaf').

Grothendieck toposes were introduced by Grothendieck in the early 60's as a generalization of sheaves on a topological space for the needs of Algebraic Geometry.

Elementary toposes were introduced by Lawvere and Tierney in 69/70. Besides achieving a remarkable simple set of axioms for Grothendieck toposes, they freed these from set-theoretical problems arising at the threshold of the theory. In Lawvere spirit, elementary topos were a first step towards the construction of rigourous foundations for Synthetic Differential Geometry and, eventually, for Continuum Mechanics. (See window 4.3.1)

The interested reader may find further developments in [1], [8], [12], [16], [30] and [29].

7 Adjointness in posets

7.1 General theory

One of the fundamental notions of category theory is that of adjoint functors. Particular cases of adjoint functors between posets appeared in the 30's in the work of Birkhoff, Tarski, Stone and others under the name of Galois connections. We will review this material as an introduction to the general notion of adjointness.

Before going into the question of adjointness, it is useful to discuss duality in posets. This notion reduces by half the proof of some statements: the other half follows by duality. (More optimistically, one can also say that it multiplies by two the number of theorems: each theorem has its corresponding dual).

Let P be a poset. We define the *dual* of P, P^{op}, to be the poset with the same underlying set, but with reverse ordering: $x \leq^{op} y$ iff $x \geq y$. Similarly, if $f : P \longrightarrow Q$ is an order-preserving map, we let $f^{op} : P^{op} \longrightarrow Q^{op}$ be the map defined by $f^{op}(x) = f(x)$. This map is again order-preserving. In other words, as rules, f^{op} and f coincide, although they have different domains and codomains. As examples, take a $\vee$-lattice P. We have an order-preserving

map $\vee : P \times P \longrightarrow P$. Then $\vee^{op} : P^{op} \times P^{op} \longrightarrow P^{op}$ is the operation $\wedge$ in P^{op}, i.e., $\vee^{op} = \wedge$. This results from the fact that $(P \times Q)^{op} = P^{op} \times Q^{op}$ and the following equivalences which amounts to prove the adjunction rule for $\wedge$ in P^{op}:

$$\frac{\dfrac{\dfrac{c \leq^{op} (a \vee b)}{a \vee b \leq c}}{a \leq c\ ,\ b \leq c}}{c \leq^{op} a\ ,\ c \leq^{op} b}$$

The reader may check some further duals: $\wedge^{op} = \vee$ as well as duality for infinitary versions of these operations: $\bigvee^{op} = \bigwedge$ and $\bigwedge^{op} = \bigvee$. Notice also that the dual of 0 (the smallest element) is 1 (the largest element) while the dual of 1 is 0, etc. Since duality is idempotent in the sense that $(P^{op})^{op} = P$ and $(f^{op})^{op} = f$, only half of these statements need proofs.

Let P and Q be two posets and let f and g be order-preserving functions as shown in the diagram

$$\begin{tikzcd} P \arrow[r, shift right, "g"'] & Q \arrow[l, shift right, "f"'] \end{tikzcd}$$

We define $f \dashv g$ to be the following relation that holds between f and g:

$$\forall x \in P\ \forall y \in Q\ \frac{f(y) \leq x}{y \leq g(x)}$$

When this relation holds, we say that f is a *left adjoint* of g or, equivalently, that g is a *right adjoint* of f.

This notion generalizes that of f and g being inverse of each other. Somehow, g is the closest thing to an inverse of f, when this doesn't exist.

Notice that $f \dashv g$ iff $g^{op} \dashv f^{op}$. This fact will be used tacitly in the sequel.

Just as inverses, adjoint maps are unique:

Proposition 7.1.1 *Adjoints maps are unique:*

(1) If f_1 and f_2 are left adjoints of g, then $f_1 = f_2$.

(2) If g_1 and g_2 are right adjoints of f, then $g_1 = g_2$.

Proof.
It suffices to prove *(1)* and use duality. But *(1)* follows from the following equivalences (given by adjointness):

$$\frac{\dfrac{f_1(y) \leq x}{y \leq g(x)}}{f_2(y) \leq x}$$

In fact, put $x = f_1(y)$. Since the first line is (trivially) true, the last must be true: $f_2(y) \leq f_1(y)$. Similarly, puting $x = f_2(y)$ and proceeding from the last to the first, $f_1(y) \leq f_2(y)$. Since we are in a poset, $f_1(y) = f_2(y)$. But y was arbitrary and hence, the conclusion follows.

Remark 7.1.2 Uniqueness of adjoints is clear from the following formulas obtained from the adjunction $f \dashv g$:

$$f(y) = \bigwedge\{x | y \leq g(x)\}$$

$$g(x) = \bigvee\{y | f(y) \leq x\}$$

These formulas will be used tacitly in the sequel.

One of the reasons that adjoint maps are so useful is the following

Proposition 7.1.3

(1) If $f : Q \longrightarrow P$ has a right adjoint, then it preserves the (existing) sups in Q.

(2) If $g : P \longrightarrow Q$ has a left adjoint, then it preserves the (existing) infs in P.

Proof.
Let us prove the first, since the other follows from the first by duality. Assume that $(y_i)_i$ is a family of elements of Q having a sup, say y. We claim that $f(y)$ is the sup of the family $(f(y_i))_i$. But this follows from the following equivalences:

$$\frac{\dfrac{\dfrac{\forall i\ f(y_i) \leq x}{\forall i\ y_i \leq g(x)}}{y \leq g(x)}}{f(y) \leq x}$$

Adjointness is a statement of equivalence: $f(y) \leq x$ is equivalent to $y \leq g(x)$ for all x and for all y. In particular, for $x = f(y)$, it follows that $y \leq g(f(y))$. Similarly, for $y = g(x)$, we obtain that $f(g(x)) \leq x$. These two particular cases suffice to obtain adjointness:

Proposition 7.1.4 *Let*

$$P \underset{g}{\overset{f}{\rightleftarrows}} Q$$

be order-preserving maps. Then the following are equivalent:

(1) $f \dashv g$

(2) $y \leq g(f(y))$ *and* $f(g(x)) \leq x$

Proof.
(1) → *(2)*: already proved in the remarks preceding the proposition.
(2) →*(1)*: let $f(y) \leq x$. Applying the (order-preserving) map g,

$$g(f(y)) \leq g(x)$$

Using the first inequality of *(2)*, we conclude that $y \leq g(x)$. In the same way (using the first inequality of (2) and g instead of f) we can conclude that $f(y) \leq x$ from $y \leq g(x)$.

We will refer to the inequalities in *(2)* of the above proposition as the *unit* and the *counit* (respectively) of the adjunction. With this terminology, this proposition says that an adjunction is completely determined by its unit and counit.

The question of deciding whether an order preserving map $f : Q \longrightarrow P$ has an adjoint is not a trivial one. Nevertheless, there are partial results:

Proposition 7.1.5

(1) Assume that Q *is complete. Then*

f *has a right adjoint iff* f *preserves all sups in* Q.

(2) Assume that P *is complete. Then*

f has a left adjoint iff f preserves all infs in P

Proof.
It is enough to prove *(1)*; *(2)* follows from duality.
$\rightarrow$: this is proposition 7.1.3
$\rightarrow$: if f had a right adjoint g, then we would have the equivalence

$$\frac{f(y) \leq x}{y \leq g(x)}$$

Thus $g(x) = \bigvee\{y : f(y) \leq x\}$, i.e., g is completely determined. Take this as a definition, noticing that it makes sense because of the completeness of Q. We have to show the adjunction: $f(y) \leq x$ iff $y \leq g(x)$. The implication from left to right is obvious, from the definition of g. To prove the implication from right to left, let $y \leq g(x) = \bigvee\{y : f(y) \leq x\}$. Applying the sup-preserving f, we obtain $f(y) \leq \bigvee\{f(y) : f(y) \leq x\} \leq x$. $\square$

Recall that a poset P is *complete* if any subset (of P) has an infimum. Dually, P is *cocomplete* if any subset has a supremum. Notice, however, that this notion is self dual: a poset is complete iff it is cocomplete. Indeed we have the following formulas which define suprema in terms of infima and vice-versa for a subset $A \subseteq P$:

$$\bigvee A = \bigwedge\{x|\forall a \in A(a \leq x)\}$$

$$\bigwedge A = \bigvee\{x|\forall a \in A(x \leq a)$$

We conclude this section with the statement that 'adjoints compose'

Proposition 7.1.6 *Assume that in the diagram of posets and order preserving functions*

$$\begin{tikzcd} P \arrow[r, shift right, "g"'] & Q \arrow[l, shift right, "f"'] \arrow[r, shift right, "k"'] & R \arrow[l, shift right, "h"'] \end{tikzcd}$$

$f \dashv g$ and $h \dashv k$. Then $f \circ h \dashv k \circ g$.

Proof.
Left as an exercise.

7.2 Logical operations as adjoint maps

It was the discovery of Bill Lawvere that most of the significant logical operations arise as adjoints to naturally given functors. This process is hierarchical, starting with adjoints to basic functors and defining new adjoints to previously obtained ones, possibly using parameters. Some exactness conditions are specified along the way to obtain 'doctrines', which are the categorical counterparts of different 'logics'. Thus we have the doctrine of Heyting categories which is the categorical counterpart of first-order many-sorted intuitionistic logic, the doctrine of Boolean categories (the categorical counterpart of first-order many-sorted classical logic), etc. Of course there are doctrines corresponding to propositional logic: Heyting algebras, Boolean algebras, and so on.

The appeal of Lawvere's approach is the possibility of generating logical operations from the notions of set, function and functional composition, via the notion of adjunction.

We will see in detail how this works for some doctrines corresponding to propositional logics. We will look at doctrines corresponding to some many-sorted, first-order logics in section 10.2

• *Smallest and largest elements*

If P is a poset with a smallest element, we let 0 be that element. Similarly, if P has a largest element, we let 1 to be the largest element. A poset with both a smallest and a largest element is called *bounded*.

We show that the existence of a smallest (and a largest) element can be described in terms of adjoints. Indeed, let $P \xrightarrow{!_P} \mathbb{1}$ be the only (vacuously sup-preserving) map from P into a poset with one element $*$.

Proposition 7.2.1

The poset P has a smallest element 0 iff $!_P$ has a left adjoint l_P.

The poset P has a largest element 1 iff $!_P$ has a right adjoint r_P. Furthermore $0 = l_P(*)$ and $1 = r_P(*)$.

Proof.
Left to the reader.

• *Infima ($\wedge$) and suprema ($\vee$)*

The notions of a $\wedge$-lattice and of a $\vee$-lattice can be defined similarly. Let $P \xrightarrow{\Delta_P} P \times P$ be the diagonal map defined by $\Delta_P(x) = (x, x)$.

Proposition 7.2.2

The poset P is an $\wedge$-lattice iff Δ_P has a right adjoint r.

The poset P is an $\vee$-lattice iff Δ_P has a left adjoint l.

Furthermore $r(x, y) = x \wedge y$ and $l(x, y) = x \vee y$.

Proof.
Let $l \dashv \Delta_P$. We have the following equivalences for x,y,z in P:

$$\frac{\frac{\frac{l(x,y) \leq z}{(x,y) \leq \Delta_P(z)}}{(x,y) \leq (z,z)}}{x \leq z \ , \ y \leq z}$$

But this shows that l satisfies the adjunction rule for $\vee$. Thus, $l(x, y) = x \vee y$. Similarly, if Δ_P has a right adjoint r, then $r(x, y) = x \wedge y$. Further details are left to the reader. □

To summarize:

Adjunction	$l \dashv \Delta_P$	$\Delta_P \dashv r$
Adjoint	$l(x, y) = x \vee y$	$r(x, y) = x \wedge y$
Unit	$(y, z) \leq (y \vee z, y \vee z)$	$x \leq x \wedge x$
Counit	$x \vee x \leq x$	$(y \wedge z, y \wedge z) \leq (y, z)$

• *Implication in an $\wedge$-poset*

Let P be an $\wedge$-poset, i.e., a poset such that $\Delta_P : P \longrightarrow P \times P$ has a right adjoint r. We know from the previous proposition that $r(x, y) = x \wedge y$. An *implication* is a function $\rightarrow : P \times P \longrightarrow P$ satisfying the condition

$$z \leq x \rightarrow y \text{ iff } z \wedge x \leq y$$

for every $x, y, z \in P$.

It is easily checked that the implication, if it exists is unique.

We shall prove that implication may be obtained as an adjoint of $\wedge$, by using parameters. Indeed, let $a \in P$ and consider the map

$$I_a : P \longrightarrow P$$

defined by $I_a(y) = y \wedge a$.

Proposition 7.2.3 *The poset P has an implication iff $\forall a \in P$ I_a has a right adjoint.*

Proof.
Assume that P has an implication. Then it is easy to check that R_a defined by $R_a(y) = a{\rightarrow}y$ is the right adjoint of I_a. To show the converse, assume that I_a has a right adjoint R_a for all $a \in P$. Therefore, we have the equivalence valid for all $a, y, z \in P$:

$$\frac{I_a(z) \leq y}{z \leq R_a(y)}$$

Thus $R_a(y) = a{\rightarrow}y$, since it satisfies the condition that defines the implication.

• *Substraction in a $\vee$-poset.*

Let P be a $\vee$-poset. Define a *substraction* or *co-implication* to be a map $\setminus : P^2 {\longrightarrow} P$ satisfying the condition

$$y \setminus x \leq z \text{ iff } y \leq x \vee z$$

for every $x, y, z \in P$.

Once again, the substraction is unique, when it exists.

Proceeding dually to the above, we define

$$S_a : P {\longrightarrow} P$$

by the formula $S_a(y) = a \vee y$

Proposition 7.2.4 *The poset P has a substraction iff $\forall a \in P$ S_a has a left adjoint.*

Proof.
Left to the reader.

We notice that a poset with implication has automatically a largest element $1 = x{\rightarrow}x$, whereas a poset with substraction has a smallest element $0 = x \setminus x$.

`EXERCISE 7.2.1`

Show in detail that implication and substractions are duals to each other.

- *Modal operators*

Modal operators can also be defined in terms of adjoint functors. Let $M \overset{i}{\hookrightarrow} P$ be an injective order-preserving map between posets. We shall assume that i has both a left and a right adjoint: $p \dashv i \dashv n$. In terms of these we may define endo operators on P: $\Diamond = ip$ and $\Box = in$. We think of P as 'the set of propositions (with the logical relation of one following from the other)', M as 'modally closed propositions', p and n as the 'most adjusted' ways to turn a proposition (maybe contingent) into a modally closed one. Then $\Diamond$ and $\Box$ may be thought of as the operators of 'possibility' and 'necessity', respectively, as suggested by

Proposition 7.2.5 *The following relations hold:*

(1) $\Box \leq 1_P \leq \Diamond$

(2) $\Box^2 = \Box\ ,\ \Diamond^2 = \Diamond$

(3) $\Diamond\Box = \Box\ ,\ \Box\Diamond = \Diamond$

(4) $\Diamond \dashv \Box$

Before going into the proof, we first prove a simple

Lemma 7.2.6 *Let*

$$P \underset{g}{\overset{f}{\leftrightarrows}} Q$$

be a couple of posets and order-preserving maps such that $f \dashv g$*. Then* $gfg = g$

Proof.
By multiplying the unit $1_Q \leq gf$ on the right by g, we obtain $g \leq gfg$. By multiplying the counit $fg \leq 1_P$ on the left by g, $gfg \leq g$. $\Box$

In particular, if g is injective, $fg = 1_P$. By duality we conclude that if $g \dashv h$ and g is injective, then $hg = 1_P$.

Proof (of the proposition).
The inequalities of *(1)* are just the counit of the adjunction $i \dashv n$ and the unit of the adjunction $p \dashv i$. On the other hand,*(2)* and *(3)* are consequences of the lemma. E.g., $\Box^2 = (in)(in) = i(ni)n = in = \Box$. Finally, *(4)* follows from *(1)* and *(3)*: the unit of the adjoints is $1_P \leq \Diamond = \Box\Diamond$, while the counit is $\Diamond\Box = \Box \leq 1_P$.

We give some examples of modal operators:
• • *Sets and downsets.* Let Q be a poset, M the poset of downsets (i.e., a set S such that whenever $x \in S$ and $y \leq x$, then $y \in S$), P the set of (arbitrary) subsets of the underlying set Q and i the inclusion. Then i has both a left and a right adjoint: $p \dashv i \dashv n$. These may be computed from the adjunction relation to obtain:

$$\begin{aligned} p(X) &= \text{(the smallest } A \in M \text{ such that } X \subseteq i(A)) \\ &= \bigcup\{\downarrow a : a \in X\} \\ n(X) &= \text{(the largest } A \in M \text{ such that } i(A) \subseteq X) \\ &= \bigcup\{\downarrow a : \downarrow a \subseteq X\} \end{aligned}$$

• • *Rough sets.* Let X be a set, let R be an equivalence relation and let $[a]$ be the equivalence class of $a \in X$. We say that $A \subseteq X$ is *saturated* if $[a] \subseteq A$ whenever $a \in A$. Take M=the set of saturated subsets of X, $P = P(X)$, the set of arbitrary subsets of X and i the inclusion. Once again, i has both a left and a right adjoint $p \dashv i \dashv n$. The computation gives:

$$p(A) = \bigcup\{[x] : [x] \cap A \neq \emptyset\}$$

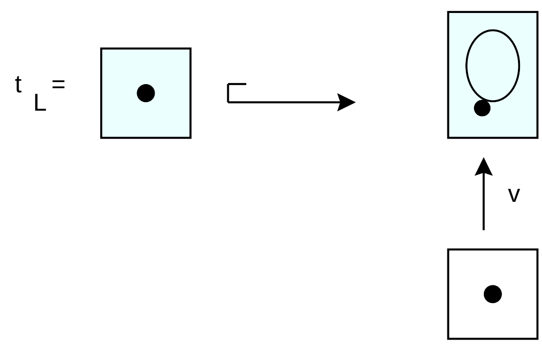
$$n(A) = \bigcup\{[x] : [x] \subseteq A\}$$

• • *Graphs and connected subgraphs.* A graph is *connected* if it is non-empy and it is not the disjoint union of two non-empty subgraphs. Take G an arbitrary graph, M= the set of connected subgraphs of G, P= the set of all the subgraphs of G and i the inclusion. Again i has both a left and a right adjoint $p \dashv i \dashv n$. The explicit computation of these adjoints is left to the reader.

Remark 7.2.7 At this point we could define and study some algebraic doctrines corresponding to modal propositional logics. The interested reader may consult [35]

• *Quantifiers*

As almost all logical operations, quantifiers can be defined by adjointness: in fact, existential and universal quantifiers turn out to be adjoints to the inverse image construction, i.e., to substitution (in logical terms). From this point of view (due to Lawvere), substitution is the basic logical operation, whereas quantifiers are obtained as adjoints. This is contrary to the usual conception among logicians, where substitution is defined from the other operations by recursion on formulas. This conception is incorrect for co-intuitionistic logic (which corresponds to co-Heyting propositional doctrine) as we saw in section 9.2. Indeed, in the diagram



we observed that $v^*(\sim t_L) \neq \sim v^*(t_L)$. Further details can be found in section 10.2.

Let $f : X \longrightarrow Y$ be a set-theoretical function. Then the inverse image of f is an order-preserving function $f^* : P(Y) \longrightarrow P(X)$, where P is the power set operation defined by $f^*(B) = \{x \in X : f(x) \in B\}$ for $B \subseteq Y$. By identifying B with the corresponding predicate defined on Y, we can write $f^*(B) = \{x \in X : B(f(x))\}$, which shows that inverse image is indeed substitution. Since inverse image preserves sups and infs and the poset of subsets of a set is complete, proposition 7.1.5 shows that f^* has both a left and a right adjoint:

$$\exists_f \dashv f^* \dashv \forall_f$$

Returning to the definition of adjoint maps, this can be stated as follows:

$$\frac{\exists_f(A) \subseteq B}{A \subseteq f^*(B)} \quad \frac{B \subseteq \forall_f(A)}{f^*(B) \subseteq A}$$

From these relations, explicit formulas may be given:

$$\exists_f(A) = \{y \in Y : \exists a \in X(f(a) = y \wedge a \in A)\}$$

$$\forall_f(A) = \{y \in Y : \forall a \in X(f(a) = y \longrightarrow a \in A)\}$$

It is easy to see that ordinary quantifiers may be obtained from

$$X \times Y \xrightarrow{\pi_1} X.$$

To summarize:

Quantifiers	Existential	Universal
Adjunction	$l \dashv f^*$	$f^* \dashv r$
Adjoint	$l = \exists_f$	$r = \forall_f$
Unit	$1_{P(X)} \leq f^*\exists_f$	$1_{P(Y)} \leq \forall_f f^*$
Counit	$\exists_f f^* \leq 1_{P(Y)}$	$f^*\forall_f \leq 1_{P(X)}$

8 Adjointness in categories

In this section we define the notion of adjunction between two functors. This notion generalizes that of adjoint maps in posets.

As in posets, it is useful to discuss first the notion of duality in categories. If $\mathbb{A}$ is a category, we define the *dual* of $\mathbb{A}$, $\mathbb{A}^{op}$, to be the category with the same objects as $\mathbb{A}$, but reverse morphisms: domain and codomain are formally reversed, but identities and compositions are preserved (although the order of composable arrows is of course reversed). We let $f^{op} : B \longrightarrow A$ in $\mathbb{A}^{op}$, be the morphism $f : A \longrightarrow B$ in $\mathbb{A}$. With this notation, the commutative triangle in $\mathbb{A}^{op}$

$$\begin{tikzcd} & C \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & B \end{tikzcd} \qquad h = g \circ f$$

is nothing but the commutative triangle in $\mathbb{A}$

$$\begin{tikzcd} & C \arrow[dl, "f^{op}"'] & \\ A & & B \arrow[ul, "g^{op}"'] \arrow[ll, "h^{op}"] \end{tikzcd} \qquad h^{op} = f^{op} \circ g^{op}$$

As in posets, if $F : \mathbb{A} \longrightarrow \mathbb{B}$ is a functor, we define $F^{op} : \mathbb{A}^{op} \longrightarrow \mathbb{B}^{op}$ to be the 'same' functor F but considered as a functor between their duals: $F^{op}(A) = F(A)$ and $F(f^{op}) = F(f)$.

As an example, assume that $\times : \mathbb{A}^2 \longrightarrow \mathbb{A}$ is a *product* functor, i.e., a functor which associates to a couple (A, B) of objects of $\mathbb{A}$ an object C such that some diagram

$$\begin{tikzcd} & A \\ C \arrow[ur] \arrow[dr] & \\ & B \end{tikzcd}$$

is a product. (For the existence of such a functor see proposition 8.2.2 and remark 8.2.3).
The reader can check that $(\mathbb{A}^{op})^2 = (\mathbb{A}^2)^{op}$ and that $\times^{op} : (\mathbb{A}^{op})^2 \longrightarrow \mathbb{A}^{op}$ is a coproduct functor. This is only one of a large list of dual notions that will be used, usually tacitly, as we proceed.

8.1 Adjoint functors

Let $\mathbb{A}$ and $\mathbb{B}$ be two categories. An *adjunction from* $\mathbb{B}$ *to* $\mathbb{A}$ is a quadruple (F, G, η, ϵ), where F and G are functors:

$$\begin{tikzcd} \mathbb{A} \arrow[r, shift right, "G"'] & \mathbb{B} \arrow[l, shift right, "F"'] \end{tikzcd}$$

and $Id_{\mathbb{B}} \xrightarrow{\eta} G \circ F$, $F \circ G \xrightarrow{\epsilon} Id_{\mathbb{A}}$ are natural transformations, called *unit* and *counit* of the adjunction, that induce bijections between the morphisms

$$\frac{FB \longrightarrow A}{B \longrightarrow GA} \quad A \in \mathbb{A}, B \in \mathbb{B}$$

in the following way (see proposition 7.1.4)

$$(1) \qquad \cfrac{FB \xrightarrow{f} A}{\cfrac{B \xrightarrow{\eta_B} GFB \xrightarrow{G(f)} GA}{B \xrightarrow{G(f) \circ \eta_B} GA}} \qquad \downarrow G$$

$$
(2) \qquad \begin{array}{c}
B \xrightarrow{g} GA \\
\hline
FB \xrightarrow{F(g)} FGA \xrightarrow{\epsilon_A} A \\
\hline
FB \xrightarrow{\epsilon_A \circ F(g)} A
\end{array} \qquad \downarrow F
$$

In other words, we require the validity of the following:

$$
\begin{cases}
\epsilon_A \circ FG(f) \circ F(\eta_B) = f \\
G(\epsilon_A) \circ GF(g) \circ \eta_B = g
\end{cases} \qquad \text{Adjunction formulas}
$$

We will often write '$F \dashv G$' as an abbreviation of the statement: '(F, G, η, ϵ) is an adjunction' (leaving η and ϵ implicitly understood). Sometimes, but less often, we use this expression to abbreviate the statement 'there are natural transformations η and ϵ such that (F, G, η, ϵ) is an adjunction'. We hope that the context makes clear which one we mean and that no confusion will arise. We will read '$F \dashv G'$ as 'F is a left adjoint to G' or, equivalently, 'G is a right adjoint to F'. Finally, we say that corresponding maps in the bijection

$$
\frac{FB \longrightarrow A}{B \longrightarrow GA}
$$

are *transpose* of each other.

Some explanation is needed for the requirement that η and ϵ be natural transformations. Notice first that the bijection between morphisms

$$
\frac{FB \longrightarrow A}{B \longrightarrow GA}
$$

induces a bijection between triangles in the same column:

$$
\begin{array}{cc}
\dfrac{\begin{tikzcd} & FB \arrow[r, "f"] & A \\ FB' \arrow[ur, "F(\phi)"] \arrow[urr, "f'"'] & & \end{tikzcd}}{\begin{tikzcd} & B \arrow[r, "g"] & GA \\ B' \arrow[ur, "\phi"] \arrow[urr, "g'"'] & & \end{tikzcd}}
&
\dfrac{\begin{tikzcd} FB \arrow[r, "f"] \arrow[drr, "f'"'] & A \arrow[dr, "\psi"] & \\ & & A' \end{tikzcd}}{\begin{tikzcd} B \arrow[r, "g"] \arrow[drr, "g'"'] & GA \arrow[dr, "G(\psi)"] & \\ & & GA' \end{tikzcd}}
\end{array}
$$

Now the point is: triangles corresponding (via this bijection) to commutative triangles are commutative precisely when η and ϵ are natural.

To see this, assume that the top triangle of the first column is commutative. We claim that so is the bottom triangle (of the first column). Recalling the way the bijection was defined, we have to show that

$$G(f) \circ GF(\phi) \circ \eta_{B'} = G(f) \circ \eta_B \circ \phi.$$

But the naturality of η implies that the diagram

$$\begin{tikzcd} B \arrow[r, "\eta_B"] & GFB \\ B' \arrow[u, "\phi"] \arrow[r, "\eta_{B'}"'] & GFB' \arrow[u, "GF(\phi)"'] \end{tikzcd}$$

is commutative, i.e., $\eta_B \circ \phi = GF(\phi) \circ \eta_{B'}$ and this finishes the proof. Naturality of ϵ is used to prove that if the bottom triangle of the second column is commutative, so is the top.

Notice that if (F, G, η, ϵ) is an adjunction from $\mathbb{B}$ to $\mathbb{A}$ ($F \dashv G$), then $(G^{op}, F^{op}, \epsilon^{op}, \eta^{op})$ is an adjunction from $\mathbb{A}$ to $\mathbb{B}$: ($G^{op} \dashv F^{op}$). Furthermore, the adjunction formulas are duals. This fact will be used tacitly in the sequel and allows one to prove statements 'by duality' as in posets.

We will state some properties of the unit and counit that we will use in the sequel.

Proposition 8.1.1 *Let (F, G, η, ϵ) be an adjunction from $\mathbb{B}$ to $\mathbb{A}$. Then*

(1) $\epsilon_{FB} \circ F(\eta_B) = 1_{FB}$

(2) $G(\epsilon_A) \circ \eta_{GA} = 1_{GA}$

(3) $\eta_{GFB} = GF(\eta_B)$

(4) $\epsilon_{FGA} = FG(\epsilon_A)$

Proof.
Statements *(1)* and *(2)* are immediate from the above adjunction formulas by putting $f = 1_{FB}$ and $g = 1_{GA}$. The third statement follows from *(1)* and the commutativity (due to the naturality of ϵ) of the following diagram:

$$
\begin{tikzcd}
FGFB \arrow[r, "FGF(\eta_B)"] \arrow[d, "\epsilon_{FB}"'] & FGFGFB \arrow[d, "\epsilon_{FGFB}"] \\
FB \arrow[r, "F(\eta_B)"'] & FGFB
\end{tikzcd}
$$

In fact, $GF(\eta_B)$ corresponds to 1_{FGFB} in the bijection (defining adjoint functors) and hence it must be equal to η_{GFB}. The last statement can be proved similarly or by duality.

Just as in the case of posets, adjoints are unique, but only in the sense that they are isomorphic: there are natural transformations that compose to give identities (on either side)

Proposition 8.1.2 *Adjoints are unique up to isomorphism. More precisely, let*

$$
\mathbb{A} \underset{F_2}{\overset{F_1}{\leftleftarrows}} \mathbb{B} \textit{ and } \mathbb{A} \underset{G_2}{\overset{G_1}{\rightrightarrows}} \mathbb{B}
$$

be functors.

(1) If $F_1 \dashv G$ and $F_2 \dashv G$, then F_1 and F_2 are isomorphic

(2) If $F \dashv G_1$ and $F \dashv G_2$, then G_1 and G_2 are isomorphic

Proof.
We will prove only the first, the other follows by duality. Let $(F_1, G, \eta_1, \epsilon_1)$ and $(F_2, G, \eta_2, \epsilon_2)$ be the adjunctions. Define natural transformations

$$
F_1 \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} F_2
$$

as follows: $\Phi_B = \epsilon_{1F_2B} \circ F_1\eta_{2B}$ and $\Psi_B = \epsilon_{2F_1B} \circ F_2\eta_{1B}$. We have to check that these families are indeed natural transformations. Let us check this for $(\Psi_B)_B$. Let $f : B \longrightarrow B'$ be a morphism of $\mathbb{B}$. We claim that the following diagram is commutative:

$$
\begin{tikzcd}
F_2B \arrow[r, "F_2(\eta_{1B})"] & F_2GF_1B \arrow[r, "\epsilon_{2F_1B}"] & F_1B \\
F_2B' \arrow[u, "F_2(f)"] \arrow[r, "F_2(\eta_{1B'})"'] & F_2GF_1B' \arrow[u, "F_2GF_1(f)"'] \arrow[r, "\epsilon_{2F_1B'}"'] & F_1B' \arrow[u, "F_1(f)"']
\end{tikzcd}
$$

The right square commutes because of the naturality of ϵ, whereas the left one commutes because it is obtained by applying the functor F_2 to the diagram

$$\begin{tikzcd}
B \arrow[r, "\eta_{1B}"] & GF_1B \\
B' \arrow[u, "f"] \arrow[r, "\eta_{1B'}"'] & GF_1B' \arrow[u, "GF_1(f)"']
\end{tikzcd}$$

which commutes because of the naturality of η_1.

We claim that $\Phi \circ \Psi = 1_{F_2}$ and $\Psi \circ \Phi = 1_{F_1}$. We just prove the first, the other being similar. Consider the following diagram:

$$\begin{tikzcd}
F_2B \arrow[r, "F_2(\eta_{1B})"] \arrow[d, "F_2(\eta_{2B})"'] & F_2GF_1B \arrow[r, "\epsilon_{2F_1B}"] \arrow[d, "F_2GF_1(\eta_{2B})"] & F_1B \arrow[d, "F_1(\eta_{2B})"] \\
F_2GF_2B \arrow[r, "F_2(\eta_{1GF_2B})"] \arrow[dr, "1_{F_2GF_2B}"'] & F_2GF_1GF_2B \arrow[r, "\epsilon_{2F_1GF_2B}"] \arrow[d, "F_2G(\epsilon_{1F_2B})"] & F_1GF_2B \arrow[d, "\epsilon_{1F_2B}"] \\
 & F_2GF_2B \arrow[r, "\epsilon_{2F_2B}"'] & F_2B
\end{tikzcd}$$

We claim that it is commutative and this obviously concludes the proof. Notice that the top left is commutative as a result of applying F_2 to a commutative diagram (η_1 being natural). The top and the bottom right squares commute because ϵ_2 is natural. Finally the triangle commutes since it results from applying F_2 to a triangle which commutes by proposition 8.1.1

Proposition 8.1.3 *Let*

$$\begin{tikzcd}
\mathbb{A} \arrow[r, shift right, "G"'] & \mathbb{B} \arrow[l, shift right, "F"']
\end{tikzcd}$$

be a diagram of categories and functors and let $Id_{\mathbb{B}} \xrightarrow{\eta} G \circ F$, $F \circ G \xrightarrow{\epsilon} Id_{\mathbb{A}}$ *be two natural tranformations. The following conditions are equivalent*

1. *The quadruple* (F, G, ϵ, η) *is an adjunction* ($F \dashv G$).

2. *The two composites*

$$FB \xrightarrow{F(\eta_B)} FGFB \xrightarrow{\epsilon_{FB}} FB$$

$$GA \xrightarrow{\eta_{GA}} GFGA \xrightarrow{G(\epsilon_A)} GA$$

give identities, namely, $\epsilon_{FB} \circ F(\eta_B) = 1_{FB}$ *and* $G(\epsilon_A) \circ \eta_{GA} = 1_{GA}$

Proof.
(1) →(2): This is proposition 8.1.1
(2) → (1): We have to prove the validity of the adjunction formulas. But notice that the naturality of η implies that the following diagram is commutative

$$\begin{tikzcd} B \arrow[r, "\eta_B"] \arrow[d, "g"'] & GFB \arrow[d, "GF(g)"] \\ GA \arrow[r, "\eta_{GA}"'] & GFGA \end{tikzcd}$$

In other words, $GF(g) \circ \eta_B = \eta_{GA} \circ g$. The rest of the proof is a simple computation:

$$G(\epsilon_A) \circ GF(g) \circ \eta_B = G(\epsilon_A) \circ \eta_{GA} \circ g = 1_{GA} \circ g = g$$

The other formula may be proved similarly. Alternatively, use duality. □

8.2 Examples of adjoint functors

• *Initial and terminal objects*

Let $\mathbb{C}$ be a category and let $\mathbb{C} \xrightarrow{!_{\mathbb{C}}} \mathbb{1}$ be the only functor from $\mathbb{C}$ into the category $\mathbb{1}$ (having one object $*$ and one morphism: the identity.)

Proposition 8.2.1
(1) $\mathbb{C}$ *has an initial object iff* $!_{\mathbb{C}}$ *has a left adjoint* l*.*
(2) $\mathbb{C}$ *has a terminal object iff* $!_{\mathbb{C}}$ *has a right adjoint* r*.*
Furthermore, $l(*)$ *is the initial object and* $r(*)$ *is the terminal object.*

Proof.
We prove *(1)*; *(2)* follows by duality.

Assume that $!_{\mathbb{C}}$ has a left adjoint l. Then we have natural bijections:

$$\cfrac{\cfrac{l(*) \longrightarrow C}{* \longrightarrow !_{\mathbb{C}}(C)}}{* \longrightarrow *}$$

Since there is one map from$*$ to itself, there is exactly one map from $l(*)$ to an arbitrary C, i.e., $l(*)$ is an initial object. The other direction is clear. $\square$

• *Coproducts and products*

These notions may be described conveniently in terms of the diagonal functor

$$\mathbb{C} \xrightarrow{\Delta_{\mathbb{C}}} \mathbb{C} \times \mathbb{C}$$

defined by the formula $\Delta_{\mathbb{C}}(C) = (C, C)$, with the obvious definition on morphisms.

Proposition 8.2.2

(1) $\mathbb{C}$ *has (binary) coproducts iff* $\Delta_{\mathbb{C}}$ *has a left adjoint.*

(2) $\mathbb{C}$ *has (binary) products iff* $\Delta_{\mathbb{C}}$ *has a right adjoint.*

Furthermore the right adjoint is the product and the left adjoint is the coproduct.

Proof.

Again we prove *(1)*.

Assume that $\Delta_{\mathbb{C}}$ has a left adjoint l: $l \dashv \Delta_{\mathbb{C}}$. Then we have a natural bijection

$$\frac{l(A, B) \xrightarrow{\theta} C}{(A, B) \xrightarrow{[f,g]} (C, C)}$$

Recall that $[f, g] = \Delta\theta \circ \eta_{(A,B)}$ and this implies that $f = \theta \circ i_A$ and $g = \theta \circ i_B$, where $\eta_{(A,B)} = [i_A, i_B]$.

The above bijection says that for every $A \xrightarrow{f} C$ and every $B \xrightarrow{g} C$ there is a unique $l(A, B) \xrightarrow{\theta} C$ such that $f = \theta \circ i_A$ and $g = \theta \circ i_B$. In diagrams:

$$\begin{tikzcd}
A \arrow[dr, "i_A"'] \arrow[drr, "f"] & & \\
& l(A,B) \arrow[r, dashed, "\theta"] & C \\
B \arrow[ur, "i_B"] \arrow[urr, "g"'] & &
\end{tikzcd}$$

This says that the diagram

$$\begin{tikzcd}
A \arrow[dr, "i_A"'] & \\
& l(A,B) \\
B \arrow[ur, "i_B"] &
\end{tikzcd}$$

is a coproduct, i.e., $l = \sqcup$.

Assume that $\mathbb{C}$ has binary coproducts. The obvious thing to do is to define a functor

$$\sqcup : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$$

whose value on (A, B) is the coproduct $A \sqcup B$. Here we meet the following difficulty: coproducts are defined up to isomorphism only; there is no such a thing as *the* coproduct of two objects, only *a* coproduct of them. Thus $\sqcup$ is not a function at the level of objects, let alone a functor. We invoke the axiom of choice at this point to define an auxiliary function which for each couple of objects (A, B) chooses *one* of their coproducts:

$$\begin{tikzcd}
A \arrow[dr, "i_A"'] & \\
& C \\
B \arrow[ur, "i_B"] &
\end{tikzcd}$$

Let $\sqcup$ be the function that associates to (A, B) the corresponding object C. One can check that this function may be extended to a functor which is left adjoint to Δ. Details are left to the reader. $\square$

Remark 8.2.3 The use of the axiom of choice seems out of order in our constructive approach. It seems more natural, as emphasized by Bénabou, to define a category with coproducts by the above proposition, i.e., a category $\mathbb{C}$ has *binary coproducts* iff the functor Δ has a left adjoint l. Then one may define $\sqcup = l$. In the category of $\mathbb{C}$-sets, the difficulty pointed out above does not occur: the existence of canonical products in *Sets* (namely ordinary cartesian products) yields the existence of canonical products in $\mathbb{C}$-sets: $X \times Y$

is defined *uniquely* as the $\mathbb{C}$-set whose F-figures are the couples (σ, τ), where σ is an F-figure of X and τ an F-figure of Y. It is easy to check that this canonical product may be extended to a product functor.

A radically different approach to this problem is due to M. Makkai [32] He replaces the notions of categories, functors, etc. by those of anacategories, anafunctors, etc. which are equivalent to the usual ones in the presence of the axiom of choice. However, the new theory may be developed *without* the axiom of choice.

• *Colimits and limits*

These notions may also be describe in terms of constant functors. Let $\mathbb{I}$ be a category of indices. Consider the diagonal or constant functor

$$\mathbb{C} \xrightarrow{\Delta} \mathbb{C}^{\mathbb{I}}$$

Proposition 8.2.4

(1) $\mathbb{C}$ has colimits indexed by $\mathbb{I}$ iff Δ has a left adjoint.

(2) $\mathbb{C}$ has limits indexed by $\mathbb{I}$ iff Δ has a right adjoint.

Furthermore, the left adjoint applied to the functor $F : \mathbb{I} \longrightarrow \mathbb{C}$ is colim$F$ and the righ adjoint applied to the functor $F : \mathbb{I} \longrightarrow \mathbb{C}$ is lim$F$.

Proof.
Let us show *(1)*. Assume that $l \dashv \Delta$. Then we have the following natural bijection:

$$\frac{l(F) \xrightarrow{\phi} C}{F \xrightarrow{\Phi} \Delta C}$$

By definition of this bijection, $\Phi = \Delta\phi \circ \eta_F$. Thus $\Phi_i = \phi \circ (\eta_F)_i$. Hence the bijection says that for every natural transformation $\Phi = (\Phi_i)_i$ there is a unique ϕ such that $\Phi_i = \phi \circ (\eta_F)_i$:

$$\begin{tikzcd}
i \arrow[dd, "\alpha"'] & & F(i) \arrow[dd, "F(\alpha)"'] \arrow[dr, "(\eta_F)_i"] \arrow[drr, "\Phi_i"] & & \\
& & & l(F) \arrow[r, dashed, "\phi"] & C \\
j & & F(j) \arrow[ur, "(\eta_F)_j"'] \arrow[urr, "\Phi_j"'] & &
\end{tikzcd}$$

This says precisely that $F \xrightarrow{\eta} \Delta lF$ is a colimit (or universal cocone).

To prove the converse, we meet the same difficulty of the coproduct case and again it is solved by using the axiom of choice. Details are left to the reader. $\Box$

Remark 8.2.5 Since *Sets* has canonical colimits (see 4.2), this is so for $\mathbb{C}$-sets and once again this difficulty does not occur for the categories that concern us.

We say that a category $\mathbb{C}$ is *complete* iff for every (small) category $\mathbb{I}$ every functor $\mathbb{I} \longrightarrow \mathbb{C}$ has a limit. We say that it is *cocomplete* iff the dual category is complete. Equivalently, if for every (small) category $\mathbb{I}$ every functor $\mathbb{I} \longrightarrow \mathbb{C}$ has a colimit.

Remark 8.2.6 Contrary to posets, the notions of 'complete' and 'cocomplete' are not equivalent in general as the following example shows: Let Ord be the ordered class of ordinals considered as a category. Since every set of ordinals has a supremum, Ord is cocomplete. On the other hand, the empty set does not have an infimum. (Otherwise this infimum would be the largest ordinal, a contradiction.) Thus Ord is cocomplete but not complete. The dual category Ord^{op} is complete but not cocomplete.

- *Exponentials*

Exponentials may also be described in terms of adjoint functors.

Proposition 8.2.7 *The functors*

$$\mathbb{C}\text{-}Sets \underset{(\ )^X}{\overset{(\ )\times X}{\rightleftarrows}} \mathbb{C}\text{-}Sets$$

are adjoint: $(\ ) \times X \dashv (\ )^X$.

Proof.
Although we have defined $(\ ) \times X$ and $(\ )^X$ on objects, we must also define them on maps. For the first, this is obvious. For the second, let $g : Y \longrightarrow Z$ be a change of figure. We define $g^X : Y^X \longrightarrow Z^X$ on an F-figure σ of Y^X (i.e., a morphism $\sigma : F \times X \longrightarrow Y$) to be $g^X(\sigma) = g \circ \sigma$. To define the unit

$$Y \xrightarrow{\eta_Y} (Y \times X)^X$$

we must associate with an F-figure σ of Y, an F-figure of $(Y \times X)^X$, i.e. a morphism $h_F \times X \longrightarrow Y \times X$. Define $\eta_Y(\sigma) = \overline{\sigma} \times 1_X$, where $\overline{\sigma} : h_F \longrightarrow Y$ is the morphism corresponding to σ by Yoneda. Thus $(\eta_Y(\sigma))_{F'}(f, \tau) = (\sigma.f, \tau)$. We define the counit to be the evaluation (see section 4.3)

$$Y^X \times X \xrightarrow{\epsilon_Y} Y$$

i.e., $\epsilon_Y(\phi, \sigma) = \phi(1_F, \sigma)$. The fact that both unit and counit are morphisms of $\mathbb{C}$-sets is easy to check and left to the reader.

According to proposition 8.1.3, it is enough to check

$$\epsilon_{Y \times X} \circ (\eta_Y \times 1_X) = 1_{Y \times X}$$
$$(\epsilon_Y)^X \circ \eta_{Y^X} = 1_{Y^X}$$

We shall prove the second, leaving the first to the reader.

Let σ be an F-figure of Y^X . By definition of η, $\eta_{Y^X}(\sigma) = \overline{\sigma} \times 1_X$. Then $(\epsilon_Y)^X(\eta_{Y^X}(\sigma)) = \epsilon_Y \circ (\overline{\sigma} \times 1_X)$, by definition of $(\epsilon_Y)^X$.

This computation can be summarized by means of the diagram

$$\begin{tikzcd}
& Y^X \arrow[r, "\eta_{Y^X}"] & (Y^X \times X)^X \arrow[r, "(\epsilon_Y)^X"] & Y^X \\
F \arrow[ur, dashed, "\sigma"] \arrow[urr, dashed, "\overline{\sigma}\times 1_X"] \arrow[urrr, dashed, "\epsilon_Y\circ(\overline{\sigma}\times 1_X)"'] & & &
\end{tikzcd}$$

We claim that $\epsilon_Y \circ (\overline{\sigma} \times 1_X) = \sigma$.

Let (f, x) be an F'-figure of $h_F \times X$. Then

$$\begin{aligned}
(\epsilon_Y \circ (\overline{\sigma} \times 1_X))(f, x) &= \epsilon_Y((\overline{\sigma} \times 1_X)(f, x)) \\
&= \epsilon_Y(\overline{\sigma}(f), x) \\
&= \epsilon_Y(\sigma.f, x) && \text{(by Yoneda lemma)} \\
&= \epsilon_Y(\sigma \circ (h_f \times 1_X), x) && \text{(action of exponentials)} \\
&= (\sigma \circ (h_f \times 1_X))(1_{F'}, x) && \text{(by definition of } \epsilon_Y\text{)} \\
&= \sigma((h_f \times 1_X)(1_{F'}, x)) \\
&= \sigma(f, x)
\end{aligned}$$

This computation may also be summarized by a diagram:

$$\begin{tikzcd}
 & F \times X \arrow[r, "\epsilon_Y \circ (\overline{\sigma} \times 1_X)"] & Y \\
F' \arrow[ur, dashed, "(f,x)"] \arrow[urr, dashed, swap, "(\epsilon_Y \circ (\overline{\sigma} \times 1_X))(f,x) = \sigma(f,x)"] & &
\end{tikzcd}$$

As in posets, a reason for the importance of adjoint functors is the following

Theorem 8.2.8 *Let* $\mathbb{A} \stackrel{F}{\longrightarrow} \mathbb{B}$ *be a functor. Then*

(1) If G has a left adjoint, then G preserves limits

(2) If G has a right adjoint, then G preserves colimits

Proof.
We recall that G preserves colimits iff for every category $\mathbb{I}$, for every functor $\mathbb{I} \stackrel{\Phi}{\longrightarrow} \mathbb{A}$ and for every universal cocone

$$\Phi \stackrel{\eta}{\longrightarrow} \Delta A$$

the canonical cocone obtained by composing with G

$$G \circ \Phi \longrightarrow \Delta GA$$

is again universal. The definition of a functor preserving limits is obtained by duality.

Let us prove *(2)*. Then *(1)* follows by duality.

Assume that G has a right adjoint R: $G \dashv R$. Let $\mathbb{I} \stackrel{\Phi}{\longrightarrow} \mathbb{A}$ be a functor and let

$$\Phi \stackrel{\eta}{\longrightarrow} \Delta A$$

be a universal cocone. We have to show that the cocone

$$G \circ \Phi \xrightarrow{\quad G \circ \eta \quad} \Delta GA$$

is universal. In fact, let $G \circ \Phi \stackrel{\theta}{\longrightarrow} \Delta GB$ be an arbitrary cocone. Consider the diagram

$$
\begin{tikzcd}
G\Phi(i) \arrow[r, "G(\eta_i)"] \arrow[dr, "\Theta_i"'] & GA \\
& B
\end{tikzcd}
$$

We have to show the existence of a unique $GA \longrightarrow B$ making that diagram commutative. By the bijection defining adjointness, this diagram corresponds to a diagram:

$$
\begin{tikzcd}
\Phi(i) \arrow[r, "\eta_i"] \arrow[dr, "\Theta_i'"'] & A \\
& RB
\end{tikzcd}
$$

By the universal property of the cocone, there is a unique $A \longrightarrow RB$ making this last diagram commutative. Hence the corresponding map $GA \longrightarrow B$ makes the above diagram commutative. Since this correspondence is bijective, this map is unique. $\Box$

Although a large part of the theory of adjointness in posets may be generalized to categories, there are some results that cannot. For instance, we would expect that if $\mathbb{A}$ is a complete category and $G : \mathbb{A} \longrightarrow \mathbb{B}$ a limit-preserving functor, then G should have a left adjoint. However this is not true as the following example shows:

Let Ord^{op} be the dual of the ordered class of ordinals considered as a category. (See remark 8.2.6.) Then Ord^{op} is complete and the trivial functor $G : Ord^{op} \longrightarrow \mathbb{1}$, into the category with one object and identity clearly preserves limits. On the other hand, the existence of a left adjoint would amount to the existence of a smallest element in Ord^{op}, i.e., a largest ordinal, a contradiction.

There is a generalization under a set-theoretical restriction, which is vacuously true in the poset case. We say that the functor

$$G : \mathbb{A} \longrightarrow \mathbb{B}$$

satisfies the *solution set condition* (SSC) iff for every object B of $\mathbb{B}$ there is a *set* $\mathcal{X}_B$ of morphisms of $\mathbb{B}$ of the form $B \longrightarrow GX$ such that every morphism $B \longrightarrow GA$ may be factorized as follows:

$$\begin{tikzcd} B \arrow[r] \arrow[dr] & GA \\ & GX \arrow[u] \end{tikzcd}$$

with $B \longrightarrow GX \in \mathcal{X}_B$ and $f : X \longrightarrow A \in \mathbb{A}$. In term of this notion, we may formulate the following result without proof, see for instance [28]

Theorem 8.2.9 (Freyd Adjoint Functor Theorem) *Let $\mathbb{A}$ be complete and let*

$$G : \mathbb{A} \longrightarrow \mathbb{B}$$

be a functor. Then G has a left adjoint iff G preseves all the limits and satisfies the (SSC).

`EXERCISE 8.2.1`

(1) Show that the Yoneda functor Y preserves all limits of $\mathbb{C}$.

(2) Show that $Y\mathbb{1} \sqcup Y\mathbb{1} \neq Y(\mathbb{1} \sqcup \mathbb{1})$ by looking at Bouquets.

9 Logical operations in $\mathbb{C}$-Sets

In this section we will study *the logic* of the category of $\mathbb{C}$-sets. More precisely, we will first study logical operations on the set of sub $\mathbb{C}$-sets of a $\mathbb{C}$-set, analogous to the well-known Boolean operations on the subsets of a set. Their properties will be specified in terms of *adjunction rule*, formally similar to the rules of inference introduced by Gentzen in Proof Theory. It turns out that, similarly to the case of sets, some of these operations may be derived from *truth functions* or *logical operations* on Ω. By these we mean morphisms $\Omega^n \longrightarrow \Omega$. But, contrary to the case of sets, there are logical operations that cannot be derived in such a way. Afterwards, we will study quantifiers. These appear as adjoints to pullbacks. Finally, we 'internalize' these notions and operations, giving examples of 'internal power sets' which play the same role in $\mathbb{C}$-sets as ordinary power sets in set theory.

9.1 The poset of sub-$\mathbb{C}$-sets of a $\mathbb{C}$-set

Let X be a $\mathbb{C}$-set. We define $\mathrm{Sub}(X)$ to be the poset of sub $\mathbb{C}$-sets of X with partial order defined by $A \leq B$ iff every F-figure of A is an F-figure of B or, what amounts to the same, if A is a sub $\mathbb{C}$-set of B.

Proposition 9.1.1 *The poset Sub(X) is a complete lattice satisfying the distributivity law:*

$$A \wedge \bigvee_i B_i = \bigvee_i A \wedge B_i$$

Proof.
Let $(A_i)_i$ be a family in $\mathrm{Sub}(X)$. Define $\bigvee_i A_i$ to be the sub $\mathbb{C}$-set of X whose F-figures are the (set-theoretical) union of the F-figures of A_i. Similarly, $\bigwedge_i A_i$ is the sub $\mathbb{C}$-set of X whose F-figures are the intersection of the F-figures of A_i. To check that these are sub $\mathbb{C}$-sets of X is trivial. It is equally trivial to check that $\bigvee_i A_i$ is the smallest sub $\mathbb{C}$-set of X greater or equal than each A_i and, similarly, that $\bigwedge_i A_i$ is the largest sub $\mathbb{C}$-set of X smaller or equal than each A_i, as befits a supremum and an infimum, respectively:

$$\frac{\bigvee_i A_i \leq B}{A_i \leq B \; \forall i} \qquad \frac{B \leq \bigwedge_i A_i}{B \leq A_i \; \forall i}$$

To prove distributivity, notice that both members of the equation are sub $\mathbb{C}$-sets of X having precisely the same F-figures, because of the corresponding distributive law for sets.

Thus, the poset $\mathrm{Sub}(X)$ is a frame in the sense of the following

Definition 9.1.2 *A frame is a complete lattice satisfying the following distributive law:*

$$a \wedge \bigvee_i b_i = \bigvee_i a \wedge b_i$$

Notice that a frame is a bounded distributive lattice: 0 is the sup of the empty family and 1 is the inf of the empty family. In our case, $0 = \emptyset$ and $1 = X$. The following is well-known, but we will sketch a proof for completeness' sake:

Proposition 9.1.3 *Every frame is a Heyting algebra.*

Proof.
We recall that a Heyting algebra is a bounded distributive lattice with a binary operation $\to$, the implication, satisfying the following *adjunction rule*:

$$\frac{c \leq a \to b}{c \wedge a \leq b}$$

Assume that such an operation exists. Then $a \to b$ is the largest of all c such that $c \wedge a \leq b$. In other words, the definition of $\to$ is forced:

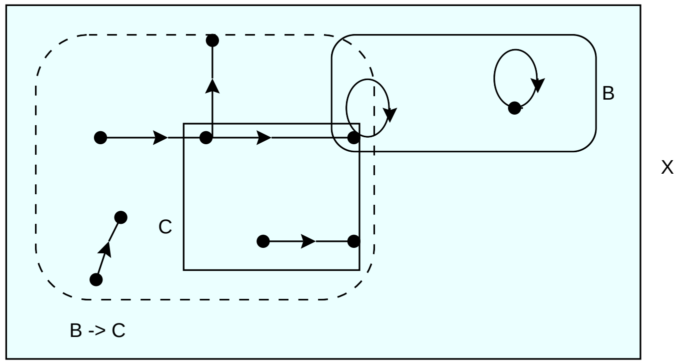
$$a \to b = \bigvee \{c : c \wedge a \leq b\}$$

Taking this formula as the definition of $\to$, we have to show the adjunction rule.
($\uparrow$): Obvious
($\downarrow$): Use the law of distributivity. $\square$

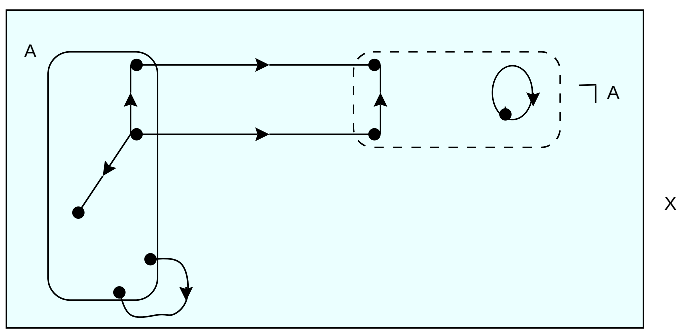
Corollary 9.1.4 *The poset $Sub(X)$ is a Heyting algebra.*

Examples of $\to$, $\neg$ and $\neg\neg$ in graphs
In the following the results of these operations are represented as dotted sub-graphs.

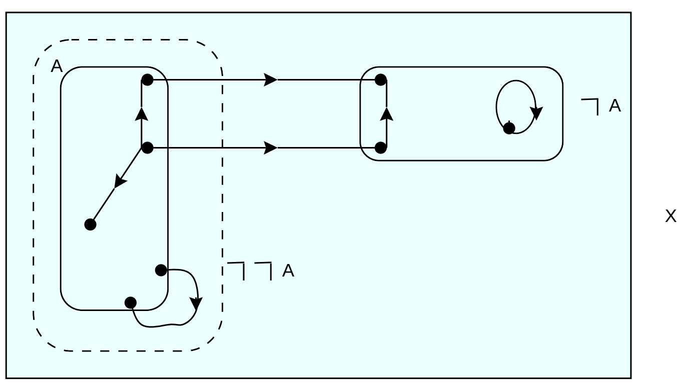
• $\rightarrow$:



• $\neg$:



• $\neg\neg$:



We now describe the operations $\wedge$, $\vee$, $\rightarrow$ and $\neg$ on Sub(X) in terms of F-figures by using the notion of *forcing*. If $A \in Sub(X)$ and σ is an F-figure of X, we shall use the notation '$F \Vdash A[\sigma]$' to mean that σ is an F-figure of A.

Proposition 9.1.5 *Let $A, B \in Sub(X)$. Then*

$$F \Vdash (A \wedge B)[\sigma] \Longleftrightarrow F \Vdash A[\sigma] \text{ and } F \Vdash B[\sigma]$$

$$F \Vdash (A \vee B)[\sigma] \Longleftrightarrow F \Vdash A[\sigma] \text{ or } F \Vdash B[\sigma]$$

$$F \Vdash (A \to B)[\sigma] \Longleftrightarrow \forall F' \xrightarrow{f} F\ (F' \Vdash A[\sigma.f] \Rightarrow F' \Vdash B[\sigma.f])$$

$$F \Vdash \neg A[\sigma] \Longleftrightarrow \forall F' \xrightarrow{f} F\ F' \nVdash A[\sigma.f]$$

Proof.
Let us prove the last equivalence. We define A' as the sub family $(A'(F))_F$ of the family $X = (X(F))_F$ such that

$$\sigma \in A'(F) \Longleftrightarrow \forall F' \xrightarrow{f} F\ F' \nVdash A[\sigma.f]$$

We have to show that $A' \in Sub(X)$. This means that if $\sigma \in A'(F)$ then $\sigma._X f \in A'(F')$ where $F' \xrightarrow{f} F$ is a change of figure. Assume that $\sigma \in A'(F)$. We have to show that $\forall F'' \xrightarrow{g} F'\ F'' \nVdash A[(\sigma.f).g]$. Let $F'' \xrightarrow{g} F'$ be a change of figure. Then $F'' \xrightarrow{f \circ g} F$ is again a change of figure. By the definition of A' and the assumption, $F'' \nVdash A[\sigma.(f \circ g)]$. The conclusion follows from the fact that $(\sigma.f).g = \sigma.(f \circ g)$.

To show that $A' = \neg A$ it is enough to check that A' satisfies the adjunction rule for $\neg$:

$$\frac{Z \leq A'}{Z \wedge A = 0_X}$$

($\downarrow$): Let $Z \leq A'$ and $\sigma \in Z(F)$. Hence $\sigma \in A'(F)$. Taking $f = 1_F$ we obtain that $\sigma \notin A(F)$. Therefore $Z(F) \cap A(F) = \emptyset$.
($\uparrow$): Let $Z \wedge A = 0_X$ and $\sigma \in Z(F)$. Therefore forall change of figure $F' \xrightarrow{f} F\ \sigma.f \in Z(F')$. This implies that for every change of figure $F' \xrightarrow{f} F$ $\sigma.f \notin A(F')$, i.e., $\sigma \in A'(F)$. □

Needless to say, the equivalences of proposition 9.1.5 may be formulated directly, without using the forcing relation and we shall often do so in the sequel. As an example, the last equivalence may be formulated as follows:

$$\sigma \in \neg A(F) \textit{ iff } \forall F' \xrightarrow{f} F\ \sigma.f \notin A(F')$$

We have used the forcing relation since it is well-known in Kripke models and Axiomatic Set Theory and it will be useful to discuss quantifiers.

WINDOW 9.1.1

Locales or spaces without points

Locales are a common generalization of topological spaces and infinitary propositional theories. However, topology has been the main source of insights and examples for the development of the theory and we start from there to motivate the main definitions.

Let X be a topological space and let $O(X)$ be the set of opens of X. Thus these contain X (the whole space), $\emptyset$ (the empty set) and are closed under $\cap$ (binary intersections) and $\bigcup$ (arbitrary unions). Furthermore, the opens satisfy the (infinite) distributive law

$$U \cap \bigcup_i U_i = \bigcup_i (U \cap U_i)$$

A continuous map $f : X \longrightarrow Y$ gives rise, by inverse image, to a map $f^{-1} : O(Y) \longrightarrow O(X)$ which sends Y into X, $\emptyset$ into $\emptyset$ and preserves $\cap$ and $\bigcup$. The basic idea that led to the theory of locales is, in a nutshell, to replace the topological space X by the frame $O(X)$ and a continuous map $f : X \longrightarrow Y$ by the frame map f^{-1}. This passage from topology to algebra, however, *changes the variance*, since f^{-1} goes in the opposite direction of f. But we can put things right by *formally* inverting arrows, i.e., by taking the dual of this category. We obtain thus the category of *locales* or *formal spaces.* The main point is that this category has better properties than the original category of topological spaces and continuous functions. Furthermore, it includes (duals of) frames that do not come from any topological space and, indeed, that in a very precise sense do not have points.

This theory has applications not only in topos theory and categorical logic, but in computer science. The interested reader may consult [11] and [13].

In the corollary 9.1.4, we proved that Sub(X) is a frame, i.e., a complete lattice satisfying the distributive law:

$$a \wedge \bigvee_i b_i = \bigvee_i a \wedge b_i$$

However, something more is true. To state this further property we need a

Definition 9.1.6 *A coframe is a poset whose dual is a frame.*

Let us spell this definition. Recall that the dual $\mathbb{P}^0$ of a poset $\mathbb{P}$ has the same underlying set, but the reverse ordering: $a \leq^0 b$ iff $a \geq b$. Since the dual of a complete lattice is again complete, or equivalently a bounded lattice having arbitrary sups has also arbitrary infs (computed by $\bigwedge S = \bigvee \{x : x \leq s \ \forall s \in S\}$), $\mathbb{P}$ is a coframe when $\mathbb{P}^0$ satisfies the above distributive law. Returning to the original poset, we can reformulate the definition as follows:

Definition 9.1.7 *A coframe is a completely distributive lattice satisfying the following law:*

$$a \vee \bigwedge_i b_i = \bigwedge_i a \vee b_i$$

To formulate some results at the right level of generality, we recall a

Definition 9.1.8 *A co-Heyting algebra is a poset whose dual is a Heyting algebra*

Equivalently, by spelling the definition as above,

Definition 9.1.9 *A co-Heyting algebra is a bounded distributive lattice with a binary operation $\backslash$ satisfying the adjunction rule*

$$\frac{a \backslash b \leq c}{a \leq b \vee c}$$

We define $\sim a = 1 \backslash a$. As a consequence, we have the following adjunction rule for $\sim$:

$$\frac{\sim a \leq c}{1 = a \vee c}$$

We have a relation between coframes and co-Heyting algebras just as that between frames and Heyting algebras:

Proposition 9.1.10 *Every coframe is a co-Heyting algebra*

Proof.
By duality or directly checking that (the forced) definition

$$a \backslash b = \bigwedge \{c : a \leq b \vee c\}$$

satisfies the adjunction rule, by using the law of distributivity of a coframe.

Proposition 9.1.11 *$Sub(X)$ is a coframe.*

Proof.
We have to prove that $(Sub(X), \leq)$ satisfies the law :

$$A \vee \bigwedge_i B_i = \bigwedge_i A \vee B_i$$

To prove that these sub $\mathbb{C}$-sets of X are equal, it is enough to prove that they have the same F-figures, for all F. But this is a well-known (and easy to prove) set-theoretical statement.

Corollary 9.1.12 *The poset $Sub(X)$ is a co-Heyting algebra.*

Corollary 9.1.13 *The poset $Sub(X)$ is a bi-Heyting (Heyting and co-Heyting) algebra.*

EXERCISE 9.1.1
Show that the frame $Sub(X)$ is generated by the irreducibles. Furthermore, show that Heyting and co-Heyting operations in $Sub(X)$ may be described in terms of irreducibles as follows. If p is an irreducible,

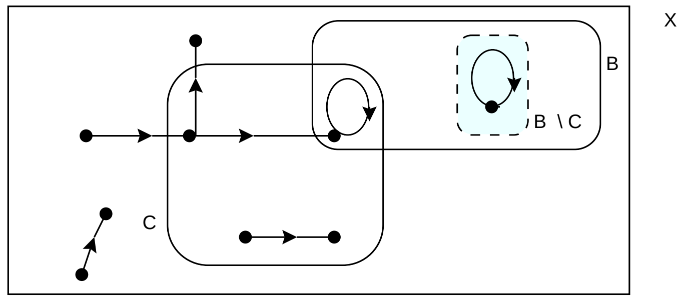
$$p \leq (A \longrightarrow B) \text{ iff } p \leq A \Rightarrow p \leq B$$

$$p \leq (A \setminus B) \text{ iff } p \leq A \wedge p \not\leq B$$

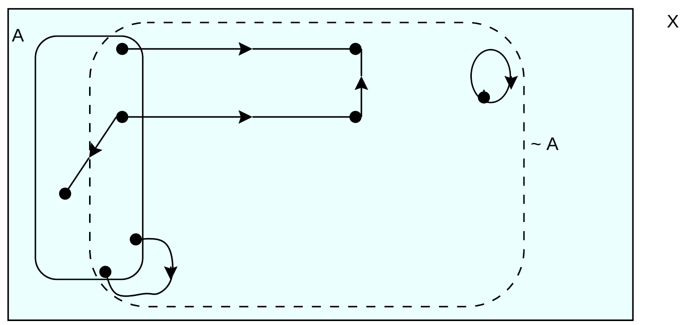
(see [35])

Examples of $\backslash$, $\sim$ *and* $\sim\sim$ *in graphs*
In the following the results of these operations are represented as dotted sub-graphs.

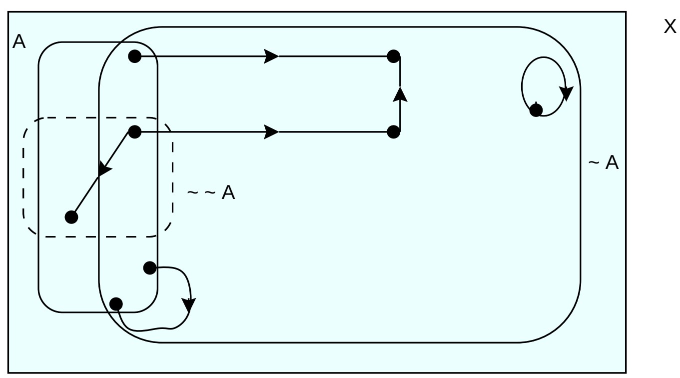
- $\backslash$:



- $\sim$:



- $\sim\sim$:



In terms of the forcing relation, we can prove

$$F \Vdash A \backslash B[\sigma] \Longleftrightarrow \exists (F \xrightarrow{f} F')\ \exists \tau \in X(F')\ F' \Vdash A[\tau], F' \nVdash B[\tau] \text{ and } \tau.f = \sigma$$

$$F \Vdash \sim A[\sigma] \Longleftrightarrow \exists (F \xrightarrow{f} F')\ \exists \tau \in X(F')\ F' \nVdash A[\tau] \text{ and } \tau.f = \sigma$$

(A F'-figure τ such that $\tau.f = \sigma$ corresponds to the notion of *counterpart* in modal logic.)

WINDOW 9.1.2

Negations in natural languages

Since antiquity two different negations in natural languages have been recognized: predicate negation (*not honest*) and predicate term negation (*dishonest*). Morphologically, the first negates the predicate *to be honest*, the second the adjective *honest*. Intuitively, predicate term negation (*dishonest*) is stronger than predicate negation (*not honest*). Even languages like Chinese, which lack morphological means to express some of these negations still have contrasts, as in English, between such predicates as *good* and *bad*, *healthy* and *sickly*, the second adjective of each pair being stronger than the predicate negation of the first.

In [18] the authors propose a model of presheaves for the two negations which allows one to discuss objectively the connections between the two negations as well as the connections between negations of predicates and negations of propositions (by means of pull-back functors). The basic idea can be stated very simply.

Think of a discussion about John's honesty. For the discussion to succeed there must be agreement about *the aspects* of John life that are considered important to decide the issue. For instance, there might be agreement that only three aspects of John's life need to be considered, those relating to his family life, his work and to other people's property. Of course, each of these aspects could be divided into sub-aspects.

This suggests to consider a small category whose generic figures are the aspects under consideration, including a global one, which is the aspect of the verdict. We say that 'John is dishonest' under a certain aspect F iff it fails to be honest under every sub-aspect of F. We say that 'John is not honest' under an aspect F iff it fails to be honest under a super-aspect of F. Comparing with the forcing relation, this amounts to interpret 'dishonest' as $\neg$*honest* and 'not honest' as $\sim$ *honest*.

The interested reader can find more information in the paper quoted.

Category-theoretic methods in semantics of natural languages and cognition have been used in the following: [26], [17], [18], [19], and [31].

9.2 Naturality of logical operations

The following shows that Heyting operations are natural

Proposition 9.2.1 *The operations $\wedge_X$, $\vee_X$, $\rightarrow_X$ and $\neg_X$ on Sub(X) are natural in the sense that the following diagrams are commutative for every morphism $Y \xrightarrow{\Phi} Z$ of $\mathbb{C}$-sets :*

$$\begin{tikzcd}
Sub(Z) \times Sub(Z) \arrow[r, "\wedge_Z(\vee_Z{,}\rightarrow_Z)"] \arrow[d, "\Phi^* \times \Phi^*"'] & Sub(Z) \arrow[d, "\Phi^*"] \\
Sub(Y) \times Sub(Y) \arrow[r, "\wedge_Y(\vee_Y{,}\rightarrow_Y)"'] & Sub(Y)
\end{tikzcd}$$

$$\begin{tikzcd}
Sub(Z) \arrow[r, "\neg_Z"] \arrow[d, "\Phi^*"'] & Sub(Z) \arrow[d, "\Phi^*"] \\
Sub(Y) \arrow[r, "\neg_Y"'] & Sub(Y)
\end{tikzcd}$$

Proof.
Let us show, for instance, that $\neg_X$ is natural. We have to show that for $B \in Sub(Y)$

$$\neg_Z \Phi^*(B) = \Phi^*(\neg_Y B)$$

But

$$\begin{aligned}
\sigma \in \Phi^*(\neg_Y B)(F) &\Leftrightarrow \Phi(\sigma) \in (\neg_Y B)(F) \\
&\Leftrightarrow \forall (F' \xrightarrow{f} F)\ \Phi(\sigma).f \notin B(F')
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sigma \in \neg_Z \Phi^*(B)(F) &\Leftrightarrow \forall (F' \xrightarrow{f} F)\ \sigma.f \notin \Phi^*(B)(F') \\
&\Leftrightarrow \forall (F' \xrightarrow{f} F)\ \Phi(\sigma.f) \notin B(F')
\end{aligned}$$

and the result follows from $\Phi(\sigma).f = \Phi(\sigma.f)$. $\Box$

Theorem 9.2.2 *There exists a bijection between morphisms of $\mathbb{C}$-sets and families of functions which are natural:*

$$\frac{\Omega^n \xrightarrow{\psi} \Omega}{(Sub(X)^n \xrightarrow{\psi_X} Sub(X))_X}$$

Proof.
Let $\mathbb{C}' = Set^{\mathbb{C}^{op}}$, the category of $\mathbb{C}$-sets. Let us consider $\mathbb{C}'$ as a category of generic figures and changes of figures and apply Yoneda lemma. We will do the proof for $n = 1$. We have the bijection

$$Sub(X) \xrightarrow{\simeq} h_\Omega(X)$$

$$Y \longmapsto \chi_Y$$

whose inverse is given by

$$(X \xrightarrow{\phi} \Omega) \longmapsto \phi^*(\top).$$

We define $h_\Omega(X) \xrightarrow{\overline{\psi}_X} h_\Omega(X)$ such that the following diagram is commutative

$$\begin{tikzcd}
Sub(X) \arrow[r, "\psi_X"] \arrow[d, "\simeq"'] & Sub(X) \arrow[d, "\simeq"] \\
h_\Omega(X) \arrow[r, "\overline{\psi}_X"'] & h_\Omega(X)
\end{tikzcd}$$

It is easy to verify that the naturality of $(\psi_X)_X$ amounts to the affirmation that $\overline{\psi}_X$ preserves the action. Finally, by Yoneda lemma we have the following bijection:

$$\frac{(h_\Omega(X) \xrightarrow{\overline{\psi}_X} h_\Omega(X))_X}{\Omega \xrightarrow{\psi} \Omega}$$

where the family $(\overline{\psi}_X)_X$ is natural.

Corollary 9.2.3 *There exists operations*

$$\Omega^2 \xrightarrow{\wedge,\vee,\longrightarrow} \Omega \qquad \textit{and} \qquad \Omega \xrightarrow{\neg} \Omega$$

which are in bijective correspondance with

$$(Sub(X)^2 \xrightarrow{\wedge_X,\vee_X,\longrightarrow_X} Sub(X))_X \qquad \textit{and} \qquad (Sub(X) \xrightarrow{\neg_X} Sub(X))_X$$

Proof.
Immediate, since proposition 9.2.1 says that these families of operations are natural. □

Let us make explicit the bijection of the theorem 9.2.2 for $n = 1$. The Yoneda bijection $\psi = (\overline{\psi})_\Omega(1_\Omega)$ together with the commutativity of the diagram

$$\begin{tikzcd}
Y \in Sub(X) \arrow[r, "\psi_X"] \arrow[d, "\chi_{(\,)}"', "\simeq"] & Sub(X) \arrow[d, "\simeq"', "\chi_{(\,)}"] \\
\chi_Y \in h_\Omega(X) \arrow[r, "\overline{\psi}_X"'] & h_\Omega(X) \\
1_\Omega \in h_\Omega(\Omega) \arrow[r, "\overline{\psi}_\Omega"'] \arrow[u, "-\circ\chi_Y"] & h_\Omega(\Omega) \arrow[u, "-\circ\chi_Y"']
\end{tikzcd}$$

give us the following relations : $\overline{\psi}_X(\chi_Y) = \psi \circ \chi_Y$ (lower one) and $\chi_{\psi_X(Y)} = \overline{\psi}_X(\chi_Y)$ (upper one). Thus,

$$\chi_{\psi_X(Y)} = \psi \circ \chi_Y$$

Or equivalently

$$\psi_X(Y) = (\psi \circ \chi_Y)^*(\top)$$

This is the relation that connects ψ with $(\psi_X)_X$.

The computation of $\psi_X(Y)$ from ψ may be presented as a chain of equivalences and implications:

$$\begin{array}{c}
Y \hookrightarrow X \\ \hline
X \xrightarrow{\chi_Y} \Omega \\ \hline
X \xrightarrow{\chi_Y} \Omega \xrightarrow{\psi} \Omega \\ \hline
X \xrightarrow{\psi\circ\chi_Y} \Omega \\ \hline
(\psi \circ \chi_Y)^*(\top) \hookrightarrow X
\end{array}
\qquad
\begin{array}{c} \\ \\ \downarrow \\ \downarrow \\ \\ \end{array}$$

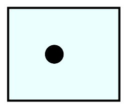
Conversely, we can compute the truth functions $\Omega^n \xrightarrow{\psi} \Omega$ from the logical operations $(Sub(X)^n \xrightarrow{\psi_X} Sub(X))_X$ by taking $\psi_F = \psi_{h_F}$ for each F and

calculating directly in the ordinary Heyting algebra $Sub(h_F) = \Omega(F)$ while forgetting the changes of figures.

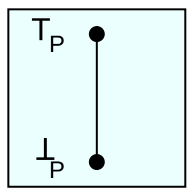
Computation of truth functions in the examples

We shall now compute some truth functions in a few examples. The interested reader may complete this list, using the same method.

• *Sets*. Let us compute $\Omega \xrightarrow{\neg} \Omega$ in detail. We start from $Sub(X) \xrightarrow{\neg_X} Sub(X)$ and we specialize to $X = h_P$, where

$h_P =$ 

From the adjunction rule for negation, $\neg_X A = \bigvee\{B : B \wedge A = 0_X\}$. Since

$Sub(h_P) =$ 

it follows that $\neg_{h_P} A = \bigvee\{B : B \wedge A = \bot_P\}$. We deduce at once the following truth-table for $\neg$:

	$\bot_P$	$\top_P$
$\neg$	$\top_P$	$\bot_P$

We now compute $\Omega \times \Omega \xrightarrow{\rightarrow} \Omega$. Once again we start from

$$Sub(X) \times Sub(X) \xrightarrow{\rightarrow_X} Sub(X)$$

and specialize for $X = h_P$. From the adjunction rule for $\rightarrow_X$,

$$A {\rightarrow_X} B = \bigvee\{C : A \wedge_X C \leq B\}$$

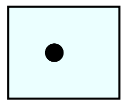
This gives at once the truth-table for $\rightarrow$:

$\rightarrow$	$\bot_P$	$\top_P$
$\bot_P$	$\top_P$	$\top_P$
$\top_P$	$\bot_P$	$\top_P$

where, as usual, the first vertical column represents the first argument and the first horizontal row represents the second argument of $\rightarrow$.

• *Bisets*. Left to the reader.

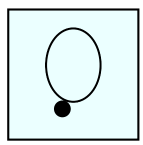
• *Bouquets*. In this case there are two generic figures: V and L. Let us start from $Sub(X) \xrightarrow{\neg_X} Sub(X)$. By specializing to $X = h_V$ where

$h_V =$ 

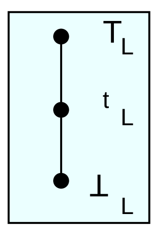
we deduce exactly as in sets the following truth-table for $\neg_V$

	$\perp_V$	$\top_V$
$\neg$	$\top_V$	$\perp_V$

To finish the description of $\neg$, let us specialise to $X = h_L$ where

$h_L =$ 

In this case

$Sub(h_L) =$ 

We may deduce the following truth-table for $\neg_L$:

	$\perp_L$	t_L	$\top_L$
$\neg_L$	$\top_L$	$\perp_L$	$\perp_L$

To deduce that $\neg_L t_L = \perp_L$, for instance, notice (by looking at the Hasse diagram) that $\perp_L$ is indeed the largest x such that $t_L \wedge x = \perp_L$.

We already proved that $\Omega \xrightarrow{\neg} \Omega$ is a morphism of bouquets, but we may check it in some cases: $(\neg_L t_L).v = \perp_L.v = \perp_V$. On the other hand,

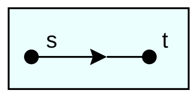
$\neg_V(t_L.v) = \neg_V(\top_V) = \bot_V$. Hence $(\neg_L t_L).v = \neg_V(t_L.v)$. Similarly$(\neg_L \top_L).v = \bot_L.v = \bot_V$. On the other hand, $\neg_V(\top_L.v) = \neg_V(\top_V) = \bot_V$. Hence $(\neg_L \top_L).v = \neg_V(\top_L.v)$.

• *Graphs*. There are two generic figures V and A. As for sets and bouquets the truth-table for $\neg_V$ is

	$\bot_V$	$\top_V$
$\neg$	$\top_V$	$\bot_V$

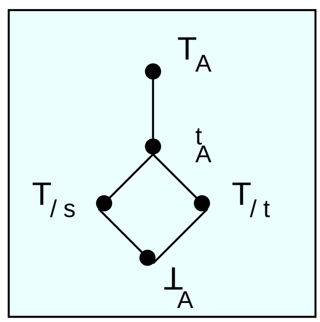
To finish the description of $\neg$, let us specialise to $X = h_A$ where

$h_A =$



In this case

Sub $(h_A) =$



We may deduce the following truth-table for $\neg_A$:

	$\bot_A$	$\top/s$	$\top/t$	t_A	$\top_A$
$\neg_A$	$\top_A$	$\top/t$	$\top/s$	$\bot_A$	$\bot_A$

since, for instance, $\top/s$ is the greatest element in the Hasse diagram disjoint from $\top/t$.

The truth-table for $\rightarrow$ consists of two parts:

$\rightarrow_V$	$\bot_V$	$\top_V$
$\bot_V$	$\top_V$	$\top_V$
$\top_V$	$\bot_V$	$\top_V$

$\to_A$	$\bot_A$	$\top/s$	$\top/t$	t_A	$\top_A$
$\bot_A$	$\top_A$	$\top_A$	$\top_A$	$\top_A$	$\top_A$
$\top/s$	$\top/t$	$\top_A$	$\top/t$	$\top_A$	$\top_A$
$\top/t$	$\top/s$	$\top/s$	$\top_A$	$\top_A$	$\top_A$
t_A	$\bot_A$	$\top/s$	$\top/t$	$\top_A$	$\top_A$
$\top_A$	$\bot_A$	$\top/s$	$\top/t$	t_A	$\top_A$

The computations are simplified using some well-known (and easy to check) results valid in every Heyting algebra: $a \to b = 1$ iff $a \leq b$; $b \leq a \to b$, etc. As an example, let us compute $(t_A \to \top/t)$ = the largest x such that $x \wedge t_A \leq \top/t$. Since $x \geq \top/t$, x is one of $\top/t$, t_A, $\top_A$. But only $x = \top/t$ verifies $x \wedge t_A \leq \top/t$. Thus, $t_A \to \top/t = \top/t$.

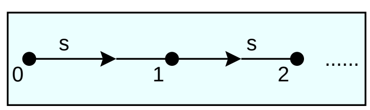
As mentioned already, we know *a priori* that $\to$ is a morphism of graphs, i.e., it preserves the incidence relations. Let us check , for instance that

$$(\top/t \to_A \top/s).s = (\top/t).s \to_V (\top/s).s.$$

Indeed, $(\top/t \to_A \top/s).s = \top/s.s = \top_V$. Similarly, $(\top/t).s \to_V (\top/s).s = (\bot_V \to_V \top_V) = \top_V$.

• *Rgraphs*. Left to the reader.

• *Esets*. In this case there is one generic figure $(*)$. By specializing to $X = h_*$ where

$h_* =$ 

and

Sub (h_*) = 0<.......<1/3<1/2<1

we deduce exactly as in the other examples the following truth-tables for $\neg_*$ and $\to_*$

	1	1/2	1/3	......	0
$\neg_*$	0	0	0	......	1

$\to_*$	1	1/2	1/3	. . .	0
1	1	1/2	1/3	. . .	0
1/2	1	1	1/3	. . .	0
1/3	1	1	1	. . .	0
.				. . .	
.				. . .	
.				. . .	
0	1	1	1	. . .	1

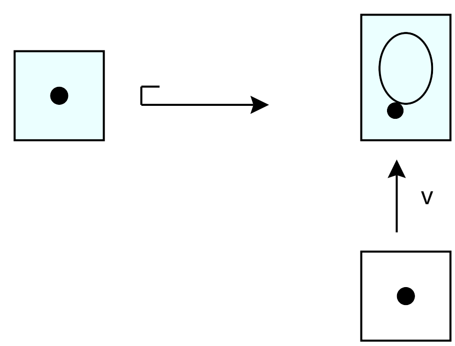
since $Sub(h_*)$ is a chain. For the same reason we have that $x \wedge y = min(x, y)$ and that $x \vee y = max(x, y)$. We can summarize the tables as follows:

$$\neg x = \begin{cases} 1 & x = 0 \\ 0 & x > 0 \end{cases}$$

$$x \to y = \begin{cases} 1 & x \leq y \\ y & x > y \end{cases}$$

Contrary to the Heyting operations, the co-Heyting operations are *not natural* in general. As an example, consider

$t_L =$



$(\sim_L\ t_L).v = \top_L.v = \top_V$ and $\sim_V\ (t_L.v) = \sim_V\ \top_V = \bot_V$. In other words, $(\sim_L, \sim_V)$ is not a morphism of bouquets and this means that $(\sim_X)_X$ is not natural and cannot be represented as an operation on Ω. In fact, we have the following

Proposition 9.2.4 *The following conditions are equivalent:*

(1) $(Sub(X) \xrightarrow{\sim_X} Sub(X))_X$ *is natural*

(2) $Sub(X)$ is a Boolean algebra, for all X

Proof.
This is a consequence of the following lemma.

Lemma 9.2.5 *Let H be a bi-Heyting algebra, i.e., Heyting and co-Heyting algebra. The following conditions are equivalent*

(i) H is a Boolean algebra

(ii) $a \wedge \sim a = 0$ for all $a \in H$

(iii) $\sim = \neg$

Proof.
Left to the reader.
Proof (of the proposition).
(1) →(2): Let us consider

$$\begin{tikzcd} \sim_X A \arrow[r, hook] & X \\ & A \arrow[u, hook, "i"'] \end{tikzcd}$$

We have that $i^*(\sim_X A) = \sim_A i^*(A)$ by hypothesis, i.e., $A \wedge \sim_X A = 0_X$ since $i^*A = A$ and the result follows from the lemma.
(2) →(1): Obvious since $\sim_X = \neg_X$ and the family $(\neg_X)_X$ is natural (see proposition 9.2.1)

There is a theorem that characterizes those categories $\mathbb{C}$ such that (2) of the previous proposition hold (for every $\mathbb{C}$-set X):

Theorem 9.2.6 (P.Freyd) *The following conditions are equivalent:*

(1) $Sub(h_F)$ is a Boolean algebra for every generic figure F

(2) $\mathbb{C}$ is a groupoid (i.e., every morphism is an isomorphism)

(3) $Sub(X)$ is a Boolean algebra for every $\mathbb{C}$-set X

Proof.
(1) $\to$ *(2)*: Let $F' \xrightarrow{f} F \in \mathbb{C}$ and consider the $\mathbb{C}$-set $< f > \leq h_F$ 'generated by f'. Namely, $g \in < f > (F'') \subseteq \mathbb{C}(F'', F)$ iff there exists h such that the following diagram commutes

$$\begin{tikzcd} & F' \arrow[r, "f"] & F \\ F'' \arrow[ur, "h"] \arrow[urr, "g"'] & & \end{tikzcd}$$

It is easy to verify that $< f > \hookrightarrow h_F$. By hypothesis, $< f > \vee \neg < f >= h_F$. In particular $1_F \in < f > (F)$ or $1_F \in \neg < f > (F)$. In the last case, using the forcing condition for negation, we obtain: $f \notin < f > (F')$, which is a contradiction. Hence, $1_F \in < f > (F)$, namely, there exists $F \xrightarrow{g} F'$ such that the following diagram commutes

$$\begin{tikzcd} & F' \arrow[r, "f"] & F \\ F \arrow[ur, "g"] \arrow[urr, "1_F"'] & & \end{tikzcd}$$

or, said otherwise, every morphism has an inverse to the right. In particular, this is true of g and so there exists $F' \xrightarrow{f'} F$ such that $g \circ f' = 1_F$. Hence $f \circ g \circ f' = f \circ 1_F = f$, thus $f' = f$ and this implies that $f \circ g = g \circ f = 1_F$.
(2) $\to$ *(3)*: Let $A \in Sub(X)$. Let us show that $A \vee \neg A = X$, namely, $A(F) \cup \neg A(F) = X(F)$. Let $\sigma \in X(F)$ such that $\sigma \notin \neg A(F)$. Therefore there is $F' \xrightarrow{f} F$ such that $\sigma.f \in A(F')$. Then $(\sigma.f).f^{-1} \in A(F)$. But $(\sigma.f).f^{-1} = \sigma.(f \circ f^{-1}) = \sigma.1_F = \sigma$.
(3) $\to$ *(1)*: Obvious. $\Box$

Remark 9.2.7

1. A detailed analysis of the proof shows that we can add the following equivalences to Freyd theorem

   (4) $\Omega(F) = 2 = \{0, 1\}$ for all generic figures F.

   (5) $Sub(X)$ is a complete and atomistic Boolean algebra (or equivalently, by Tarski's theorem, of the form 2^S, for a set S).

2. Although $\sim$ is not natural, we have, however, that it is *lax natural* in the sense that $\sim_X \Phi^*(B) \hookrightarrow \Phi^*(\sim_Y B)$ for every morphism of $\mathbb{C}$-sets $X \xrightarrow{\Phi} Y$ and every $B \in Sub(Y)$.

EXERCISE 9.2.1

(1) Give examples of $\rightarrow$, $\neg$, $\neg\neg$, $\backslash$, $\sim$ and $\sim\sim$ in bouquets and evolutive sets.

(2) Give examples of $\neg\sim$ and $\sim\neg$ in graphs, bouquets and evolutive sets.

(3) In a co-Heyting algebra we can define the *boundary* of an element a, by $\partial a = a \wedge \sim a$. Prove that the following holds in any co-Heyting algebra:

   (i) $\partial(a \wedge b) = \partial(a) \wedge b \vee a \wedge \partial(b)$

   (ii) $\partial(a \wedge b) \vee \partial(a \vee b) = \partial(a) \vee \partial(b)$

(4) (Bénabou) Let $N : \Omega \longrightarrow \Omega$ be such that $N \leq Id_\Omega$ (i.e., $N_F(\sigma) \leq \sigma \quad \forall F \in \mathbb{C},\ \sigma \in \Omega(F)$. Show that for every $\sigma \in \Omega(F) \quad N_F(\sigma) = \sigma \wedge N_F(\top_F)$

(5) Let $\leq \hookrightarrow \Omega \times \Omega$ be the sub-$\mathbb{C}$-set whose F-figures (σ, τ) are those of $\Omega \times \Omega$ such that $\sigma \leq \tau$ in $\Omega(F)$. Show that $\rightarrow = \chi_\leq$

9.3 Quantifiers

In this section we generalize quantifiers from the category of Sets (see section 7.2) to a category of $\mathbb{C}$-Sets by defining them as adjoints to pullbacks. More precisely

Proposition 9.3.1 *If $X \xrightarrow{\Phi} Y$ is a morphism of $\mathbb{C}$-Sets, then the pullback functor $Sub(Y) \xrightarrow{\Phi^*} Sub(X)$ has both a left and a right adjoint:*

$$\exists_\Phi \dashv \Phi^* \dashv \forall_\Phi$$

Proof.
Let $A \in Sub(X)$. Define the set of F-figures of $\exists_\Phi A$ to be the ordinary (set-theoretical) image of $A(F) \subseteq X(F)$ under Φ_F, i.e., $(\exists_\Phi A)(F) = \exists_{\Phi_F} A(F)$. It is easy to see that $\exists_\Phi A \hookrightarrow Y$. The following computation shows that $\exists_\Phi \dashv \Phi^*$:

$$\begin{array}{cl}
\exists_\Phi A \leq B & \\
\hline
(\exists_\Phi A)(F) \subseteq B(F) \quad \forall F \in |\mathbb{C}| & \\
\hline
\exists_\Phi A(F) \subseteq B(F) \quad \forall F \in |\mathbb{C}| & \text{by definition of } \exists_\Phi A \\
\hline
A(F) \subseteq \Phi_F^* B(F) \quad \forall F \in |\mathbb{C}| & \text{since } \exists_{\Phi_F} \dashv \Phi_F^* \\
\hline
A(F) \subseteq (\Phi_F^* B)(F) \quad \forall F \in |\mathbb{C}| & \text{by definition of } \Phi^* B \\
\hline
A \leq \Phi^* B &
\end{array}$$

By writing '$F \Vdash A[\sigma]$' for $\sigma \in A(F)$ as we did in section 9.1, we may define for every F-figure τ of Y

$$F \Vdash \exists_\Phi A[\tau] \Leftrightarrow \exists \sigma X(F)[\Phi_F(\sigma) = \tau \text{ and } F \Vdash A[\sigma]]$$

The reader may easily check that

$$(\exists_\Phi A)(F) = \{\tau \in Y(F) \mid F \Vdash \exists_\Phi A[\tau]\}$$

To show the existence of the right adjoint we cannot simply define

$$(\forall_\Phi A)(F) = \forall_{\Phi_F} A(F)$$

(See exercise 9.3.1).

We proceed instead by defining

$$F \Vdash \forall_\Phi A[\tau] \Leftrightarrow \forall F' \xrightarrow{f} F \quad \in \mathbb{C} \;\; \forall \sigma \in X(F') \;\; [\Phi_{F'}(\sigma) = \tau.f \Rightarrow F' \Vdash A[\sigma]]$$

and then

$$\forall_\Phi A(F) = \{\tau \in Y(F) \mid F \Vdash \forall_\Phi A[\tau]\}$$

Once again $\forall_\Phi A \hookrightarrow Y$. Let us show that $\Phi^* \dashv \forall_\Phi$, i.e.,

$$\frac{B \leq \forall_\Phi A}{\Phi^* B \leq A}$$

$\downarrow$: Assume that $B \leq \forall_\Phi A$ and let $\sigma \in \Phi^* B(F)$. Since $\Phi^* B(F) = \Phi_F^* B(F)$ by definition of Φ^*, it follows that $\Phi_F(\sigma) \in B(F)$, i.e., $F \Vdash B[\Phi_F(\sigma)]$.

Using the hypothesis, $F \Vdash \forall_\Phi A[\Phi_F(\sigma)]$ and this implies by definition of $\forall_\Phi$ (for $f = 1_F$) that $F \Vdash A[\sigma]$.

$\uparrow$: Assume that $\Phi^* B \leq A$ and let $F \Vdash B[\tau]$ (with $\tau \in Y(F)$), $F' \xrightarrow{f} F \quad \in \mathbb{C}$ and $\sigma \in X(F')$ such that $\Phi_{F'}(\sigma) = \tau.f$. We claim that $F' \Vdash A[\sigma]$.

Indeed, $F' \Vdash B[\tau.f]$, since $B \hookrightarrow Y$. Thus $\sigma \in (\Phi^* B)(F')$ and so $\sigma \in A(F')$ by hypothesis. In other words, $F' \Vdash A[\sigma]$. $\square$

EXERCISE 9.3.1

> Find $X \xrightarrow{\Phi} Y$ a morphism of graphs and a sub-graph A of X such that $\forall_\Phi A(F) \neq \forall_{\Phi_F} A(F)$ for some generic figure F. Hint: Take X to be a graph with two vertices x_0 and x_1, Y a graph with two vertices y_0 and y_1 and an arrow f from y_0 to y_1. Define Φ to be the inclusion map of X into Y.

For the record, we compile the forcing definitions given up to now in the following table (here $\Phi : X \longrightarrow Y,\ A,\ A_1,\ A_2 \in Sub(X)$ and $\sigma \in X(F)$):

$F \Vdash (A_1 \wedge A_2)[\sigma] \Leftrightarrow F \Vdash A_1[\sigma]$ and $F \Vdash A_2[\sigma]$
$F \Vdash (A_1 \vee A_2)[\sigma] \Leftrightarrow F \Vdash A_1[\sigma]$ or $F \Vdash A_2[\sigma]$
$F \Vdash \top[\sigma]$ always
$F \Vdash \bot[\sigma]$ never
$F \Vdash (A_1 \rightarrow A_2)[\sigma] \Leftrightarrow \forall F' \xrightarrow{f} F \ \ [F' \Vdash A_1[\sigma.f] \Rightarrow F' \Vdash A_2[\sigma.f]]$
$F \Vdash \neg A[\sigma] \Leftrightarrow \forall F' \xrightarrow{f} F \ \ [F' \nVdash A[\sigma.f]]$
$F \Vdash (A_1 \setminus A_2)[\sigma] \Leftrightarrow \exists F \xrightarrow{f} F' \ \exists \sigma' \in X(F')$ $[\sigma'.f = \sigma$ and $F' \Vdash A_1[\sigma']$ and $F' \nVdash A_2[\sigma']]$
$F \Vdash \sim A[\sigma] \Leftrightarrow \exists F \xrightarrow{f} F' \ \ \exists \sigma' \in X(F') \ \ [\sigma'.f = \sigma$ and $F' \nVdash A[\sigma']]$
$F \Vdash \exists_\Phi A[\tau] \Leftrightarrow \exists \sigma \in X(F) \ \ [\Phi_F(\sigma) = \tau$ and $F \Vdash A[\sigma]]$
$F \Vdash \forall_\Phi A[\tau] \Leftrightarrow \forall F' \xrightarrow{f} F \ \ \forall \sigma \in X(F') \ \ [\Phi_{F'}(\sigma) = \tau.f \rightarrow F' \Vdash A[\sigma]]$

Notice that $F \Vdash \neg A[\sigma]$ precisely when $F \Vdash (A \rightarrow \bot)[\sigma]$ and that $F \Vdash \sim A[\sigma]$ precisely when $F \Vdash (\top \setminus A)[\sigma]$.

EXERCISE 9.3.2

(1) (Beck-Chevalley condition) Show that whenever the diagram of $\mathbb{C}$-sets

$$\begin{tikzcd} X \arrow[r, "\Phi"] & Y \\ P \arrow[u, "\Psi'"] \arrow[r, "\Phi'"'] & Z \arrow[u, "\Psi"'] \end{tikzcd}$$

is a pullback, the diagram

$$\begin{tikzcd}
Sub(X) \arrow[r, "\exists_\Phi"] & Sub(Y) \\
Sub(P) \arrow[u, "(\Psi')^*"] \arrow[r, "\exists_{\Phi'}"'] & Sub(Z) \arrow[u, "\Psi^*"']
\end{tikzcd}$$

is commutative

(2) (Frobenius condition) Assume that $X \xrightarrow{\Phi} Y$ is a morphism of $\mathbb{C}$-sets. Show that in the diagram of adjoint functors

$$\begin{tikzcd}
Sub(X) \arrow[r, shift right, "\Phi^*"'] & Sub(Y) \arrow[l, shift right, "\exists_\Phi"']
\end{tikzcd}$$

(with $\exists_\Phi \dashv \Phi^*$) the following relation holds:

$$\exists_\Phi(\phi^*(B) \wedge A) = B \wedge \exists_\Phi(A)$$

whenever $A \in Sub(X)$ and $B \in Sub(Y)$.

9.4 Internal power sets

We have shown that the category of $\mathbb{C}$-sets has exponentials. More precisely, the endofunctor $(\ ) \times X$ has a right adjoint $(\ )^X$. We will be interested in the value of this right adjoint at Ω. In other words, we will study the object Ω^X, which we call the *internal power set of* X and show that the logical operations defined on $Sub(X)$ may be 'internalized' to Ω^X.

We define an *internal poset* to be a couple (X, R) of $\mathbb{C}$-sets, where $R \hookrightarrow X \times X$, having the property that for every generic figure F, the couple $(X(F), R(F)$ is an ordinary (i.e., set-theoretical) poset. An *internal morphism* of (internal) posets $\Phi : (X, R) \longrightarrow (Y, S)$ is a morphism of $\mathbb{C}$-sets $\Phi : X \longrightarrow Y$ such that for every generic figure F, its F-component is an ordinary order-preserving map. If

$$\begin{tikzcd}
P \arrow[r, shift right, "\Psi"'] & Q \arrow[l, shift right, "\Phi"']
\end{tikzcd}$$

is a diagram of internal posets and internal morphisms, we say that Φ *is an internal left adjoint of* Ψ, $\Phi \dashv \Psi$, if for every generic figure F their F components are adjoints in the ordinary set-theoretical sense: $\Phi_F \dashv \Psi_F$.

Proposition 9.4.1 *(1) In the diagram*

$$\Omega \underset{\wedge}{\overset{\vee}{\underset{\delta}{\leftleftarrows\!\!\rightarrow}}} \Omega\times\Omega$$

where δ is the obvious internal constant functor, the following relation holds:

$$\vee \dashv \delta \dashv \wedge$$

(2) In the diagram

$$\Omega^X \underset{\wedge^X}{\overset{\vee^X}{\underset{\delta^X}{\leftleftarrows\!\!\rightarrow}}} \Omega^X\times\Omega^X$$

the following relation holds:

$$\vee^X \dashv \delta^X \dashv \wedge^X$$

Proof.
We prove only *(1)*, leaving *(2)* to the reader.

Recall that $\Omega(F) = Sub(h_F)$. Thus the F-figures of Ω are ordered by the natural order on $Sub(h_F)$. Furthermore, there is an obvious constant map

$$\delta_F : Sub(h_F) \longrightarrow Sub(h_f) \times Sub(h_F)$$

which sends a into (a, a). As we proved in section 8 the F-component of $\vee$ is its left adjoint, whereas the F-component of $\wedge$ is its right adjoint.□

Quantifiers may also be 'internalized'. In fact we have the following

Proposition 9.4.2 *Let $\Phi : X \longrightarrow Y$ be a morphism of $\mathbb{C}$-sets. Then the morphism*

$$\Omega^\Phi : \Omega^Y \longrightarrow \Omega^X$$

has both an internal left adjoint and an internal right adjoint:

$$\exists_\Phi \vdash \Omega^\Phi \dashv \forall_\Phi$$

Proof.
Notice first that the F-figures of Ω^X are precisely the poset $Sub(h_F \times X)$. Since the F-component of Ω^Φ is the pullback $(1 \times \Phi)^*$, the F-component of its left adjoint is forced to be $\exists_{(1\times\Phi)}$ while its right adjoint must be $\forall_{(1\times\Phi)}$.

We have to verify that these adjoints are the components of a natural transformation. We shall do this for the left adjoint, leaving the other to the reader. Assume that $f : F' \longrightarrow F$ is a change of figure. We have to show that the diagram:

$$\begin{tikzcd}
\Omega^X(F) \arrow[r, "(\exists_\Phi)_F"] \arrow[d, "\Omega^X(f)"'] & \Omega^Y(F) \arrow[d, "\Omega^Y(f)"] \\
\Omega^X(F') \arrow[r, "(\exists_\Phi)_{F'}"'] & \Omega^Y(F')
\end{tikzcd}$$

is commutative.

This, however, is a consequence of the Beck-Chevalley condition applied to the pullback diagram

$$\begin{tikzcd}
h_F \times X \arrow[r, "1_F \times \Phi"] & h_F \times Y \\
h_{F'} \times X \arrow[r, "1_{F'} \times \Phi"'] \arrow[u] & h_{F'} \times Y \arrow[u]
\end{tikzcd}$$

Another important property of Ω^X : it is an internally cocomplete poset and even an internal frame. The proofs of these results will be left to the reader (see exercise 9.4.1). Here we shall prove this for Ω.

First, some definitions.

To define an *internally cocomplete poset* we return to the ordinary set-theoretical definition of a poset $(P, \leq)$ with arbitrary sups. We may formalize this notion by requiring the existence of a function

$$2^P \xrightarrow{\bigvee} P$$

which satisfies the adjunction rule

$$\frac{\bigvee A \leq a}{\forall x \in A (x \leq a)}$$

Notice that by defining $\downarrow(a) = \{x | x \leq a\}$ we may re-write the adjunction rule as

$$\frac{\bigvee A \leq a}{A \subseteq \downarrow(a)}$$

But this is the expression of an adjointness! In fact, it says that in the diagram

$$P \underset{\downarrow(\ )}{\overset{\bigvee}{\rightleftarrows}} 2^P$$

we have the adjunction relation: $\bigvee \dashv \downarrow(\ )$.

But the story does not end here. The map $\downarrow(\ )$ corresponds, by exponential adjointness, to a map $P \times P \longrightarrow 2$, which in turn is the characteristic function of some relation $G \subseteq P \times P$. Which relation? Answer: the reciprocal $\geq$ of $\leq$.

We define an *internally cocomplete poset* to be an internal poset (X, R) such that the exponential adjoint of the characteristic morphism of the reciprocal of R,

$$P \xrightarrow{\downarrow(\,)} \Omega^P$$

has an internal left adjoint.

Theorem 9.4.3 *The internal poset* $(\Omega, \leq)$ *is internally cocomplete*

Proof.
The above description of $\downarrow(\ )$ allows one to compute $\downarrow_F(\ )$. In fact,

$$\downarrow_F(a)(F') = \{(f, b) | b \leq a.f\}$$

where a is an F-figure of Ω. This forces the definition of the F-component of its left adjoint, namely

$$\bigvee_F A = \bigcup \{\exists_f(a) | (f, a) \in A(F'),\ F' \in \mathbb{C}\}$$

Unfortunately, we are not finished yet We still need to prove the following

Lemma 9.4.4 *The family* $(\bigvee_F)_F$ *is a natural transformation.*

Proof.
Let $f : F' \longrightarrow F$ be a change of figure. We have to show that the diagram

$$\begin{tikzcd}
\Omega^{\Omega}(F) \arrow[r, "\bigvee_F"] \arrow[d, "(f\times 1_\Omega)^*"'] & \Omega(F) \arrow[d, "f^*"] \\
\Omega^{\Omega}(F') \arrow[r, "\bigvee_{F'}"'] & \Omega(F')
\end{tikzcd}$$

is commutative.

In other words, we have to show that if A is an F-figure of $\Omega^{\Omega}(F) = Sub(h_F \times \Omega)$, then

$$f^*\bigvee_F A = \bigvee_{F'}(f \times 1_\Omega)^* A$$

Returning to the definition of $\bigvee$ it is enough to show the equivalence of the following statements

$$\frac{(1)\ \ \forall F'''\forall(g,a) \in A(F''')\ f^*\exists_g(a) \leq c}{(2)\ \ \forall F'''\forall h\ \forall b\ (\text{if } (fh,b) \in A(F''') \text{ then } b \leq h^*(c))}$$

$\downarrow$: Let F''', h, b be such that $(fh, b) \in A(F''')$. By (1), $f^*\exists_{fh}(b) \leq c$. Therefore $h^*f^*\exists_{fh}(b) \leq h^*(c)$. But $b \leq (fh)^*\exists_{fh}(b)$ (unit) and hence $b \leq h^*(c)$.

$\uparrow$: Let F'', $(g, a) \in A(F'')$. We have to show that $f^*\exists_g(a) \leq c$.

Let $h \in f^*\exists_g(a)(F''')$. Therefore there is a $k \in a(F''')$ such that $fh = gk$. In diagrams:

$$\begin{tikzcd}
F'' \arrow[r, "g"] & F \\
F''' \arrow[u, "k"] \arrow[r, "h"'] & F' \arrow[u, "f"']
\end{tikzcd}$$

Since $(g, a) \in A(F'')$, $(g, a).k \in A(F''')$, i.e., $(gk, a.k) \in A(F''')$. Using the fact that $fh = gk$, we conclude that $(fh, a.k) \in A(F''')$. By (2), $a.k = k^*a \leq h^*(c)$. But $a.k = F'''$ and this shows that $h^*(c) = F'''$, i.e., $h \in c(F''')\Box$

To define an internal frame we proceed as before. Recall that a set-theoretical frame is a cocomplete poset $(P, \leq)$ which satisfies the distributive law (stated in terms of subsets, rather than families)

$$b \wedge \bigvee A = \bigvee\{c | \exists a(c = b \wedge a)\}$$

The map

$$P \times 2^P \xrightarrow{K} 2^P$$

defined by $K(b, A) = \{c | \exists a \in A(c = b \wedge a)\}$ may be obtained by the following operations:

(1) Take the composite

$$P \times P \times 2^P \times P \overset{(\pi_3,\pi_4)}{\longrightarrow} 2^P \times P \overset{e}{\longrightarrow} 2$$

which sends (b, c, A, a) into $e(A, a)$, i.e., into the proposition $a \in A$

(2) Take the composite

$$P \times P \times 2^P \times P \overset{(\wedge\circ(\pi_1,\pi_4),\pi_2)}{\longrightarrow} P \times P \overset{\chi_=}{\longrightarrow} 2$$

which sends (b, c, A, a) into $\chi_=(c, b\wedge a)$, i.e., into the proposition $c = b\wedge a$

(3) Combine the previous maps into one and compose with $\wedge$ to obtain a map

$$P \times P \times 2^P \times P \longrightarrow 2$$

which sends (b, c, A, a) into $e(A, a)\wedge\chi_=(c, b\wedge a)$, i.e., into the proposition $a \in A$ and $c = b \wedge a$

(4) Take the exponential adjoint of this map to obtain the map

$$P \times P \times 2^P \longrightarrow 2^P$$

which sends (b, c, A) into $\{a | a \in A \text{ and } c = b \wedge a\}$

(5) Compose this map with the left adjoint$\exists_!$ to the pullback $!^* : 2 \longrightarrow 2^P$ where $! : P \longrightarrow 1$ is the only map into the terminal object 1, to obtain the map

$$P \times P \times 2^P \longrightarrow 2$$

which sends (b, c, A) into the proposition $\exists a(a \in A \text{ and } c = b \wedge a)$

(6) Take the exponential adjoint of this map to obtain the sought K.

We define an *internal frame* to be an internally cocomplete poset which is an internal $\wedge$-lattice such that the diagram

$$
\begin{array}{ccc}
P \times \Omega^P & \xrightarrow{K} & \Omega^P \\
{\scriptstyle 1_P \times \bigvee}\downarrow & & \downarrow{\scriptstyle \bigvee} \\
P \times P & \xrightarrow[\wedge]{} & P
\end{array}
$$

is commutative.

Remark 9.4.5 An internally cocomplete poset is automatically an internal $\wedge$-lattice. In fact, much more is true. Just as in the set-theoretical case, an internally cocomplete poset (P, R) is internally complete in the sense that the morphism

$$P \xrightarrow{\downarrow(\,)} \Omega^P$$

has an internal right adjoint.

Theorem 9.4.6 *The object Ω (with its natural order $\leq$) is an internal frame.*

Proof.
An admittedly long, although straightforward computation gives the following description for the F-component of K :

$$K_F(b, A)(F') = \{(f, c) | \exists a((f, a) \in A(F') \text{ and } c = b.f \wedge a)\}$$

Chasing the diagram which defines an internal frame, computed at the generic figure F, we are reduced to prove (dropping sub-indices) that

$$b \wedge \bigvee A = \bigvee K(b, A)$$

But $Sub(h_F)$ is a set-theoretical frame. Thus

$$
\begin{array}{lcl}
b \wedge \bigvee A & = & \bigcup\{b \wedge \exists_f(a) | (f, a) \in A(F'),\ F' \in \mathbb{C}\} \text{ and} \\
\bigvee K(b, A) & = & \bigcup\{\exists_f(c) | \exists a((f, a) \in A(F') \text{ and } c = b.f \wedge a),\ F' \in \mathbb{C}\} \\
 & = & \{\exists_f(b.f \wedge a) | (f, a) \in A(F'),\ F' \in \mathbb{C}\}
\end{array}
$$

We finish the proof by using the Frobenius condition (see exercise 9.3.2)

$$\exists_f(b.f \wedge a) = b \wedge \exists_f(a)$$

WINDOW 9.4.1

Internal language

This proof shows what is involved in the basic tool of using 'the internal language' of a category of $\mathbb{C}$-sets and, more generally, of a topos. The idea is to show that set-theoretical (or type-theoretical) statements can be interpreted in such categories and, provided that only constructive arguments are used, valid arguments remain valid in the topos. The above proof shows in a particular example how this works. For more on this, see e.g. [9] and [16]

`EXERCISE 9.4.1`

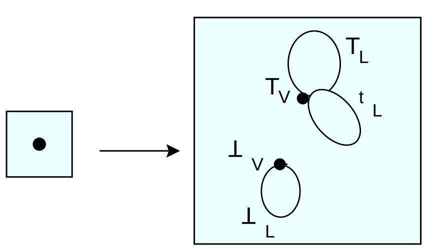
Show that Ω^X is an internal frame.

9.5 Examples of internal power sets

This section contains several examples of 'internal power sets' in the categories of bouquets and graphs. We emphasize again that Ω^X is *not* a set at all, but a $\mathbb{C}$-set.

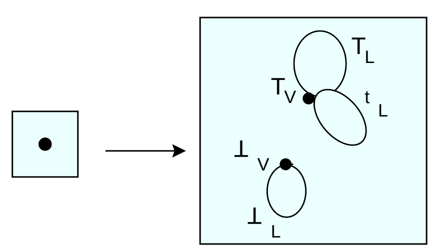
Let us start with Ω^V in bouquets:

$$\frac{\dfrac{V \longrightarrow \Omega^V}{V \times V \longrightarrow \Omega}}{V \longrightarrow \Omega}$$



$$\perp_V \; , \; \top_V$$

$$\frac{\dfrac{L \longrightarrow \Omega^V}{L \times V \longrightarrow \Omega}}{V \longrightarrow \Omega}$$



$$\perp_V \; , \; \top_V$$

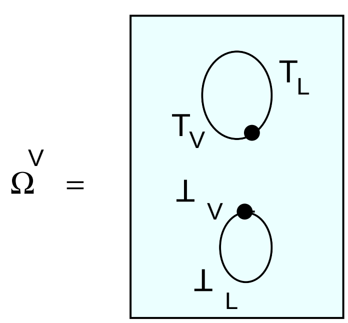
This means that there are two vertices and two loops. We must calculate the incidence relations:

$$\begin{tikzcd}
 & L \arrow[r, dashed] & \Omega^V \\
V \arrow[ur, "v"] \arrow[urr] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & L \times V \arrow[r, dashed, "\sigma"] & \Omega \\
V \times V \arrow[ur, "v\times 1_V"] \arrow[urr, "\sigma.1_V=\sigma"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & V \arrow[r, "\sigma"] & \Omega \\
V \arrow[ur, "1_V"] \arrow[urr, "\sigma.1_V=\sigma"'] & &
\end{tikzcd}$$

This means that $\Omega^V = \mathbb{1} + \mathbb{1}$ and graphically:



Comparing Ω with Ω^V we see that the loop t_L was lost in the process.

`EXERCISE 9.5.1`

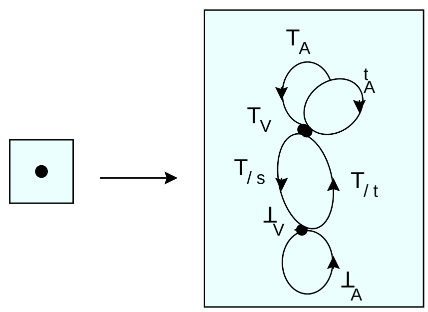
Calculate Ω^L.

Let us now calculate Ω^V in graphs.

$$V \longrightarrow \Omega^V$$

$$V \times V \longrightarrow \Omega$$

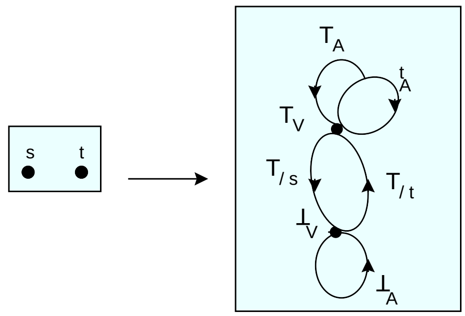
$$V \longrightarrow \Omega$$



$$\perp_V \ , \ \top_V$$

$$A \longrightarrow \Omega^V$$

$$A \times V \longrightarrow \Omega$$



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4 arrows

$s \longrightarrow \top_V$	$s \longrightarrow \bot_V$	$s \longrightarrow \top_V$	$s \longrightarrow \bot_V$
$t \longrightarrow \top_V$	$t \longrightarrow \top_V$	$t \longrightarrow \bot_V$	$t \longrightarrow \bot_V$

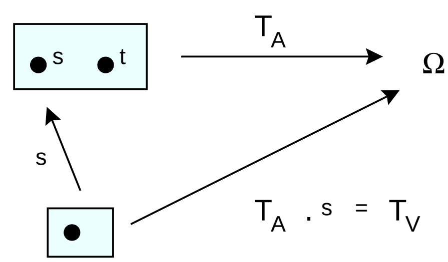
$$\top_A \qquad \top/t \qquad \top/s \qquad \bot_A$$

There are two vertices and four arrows. We must now calculate the incidence

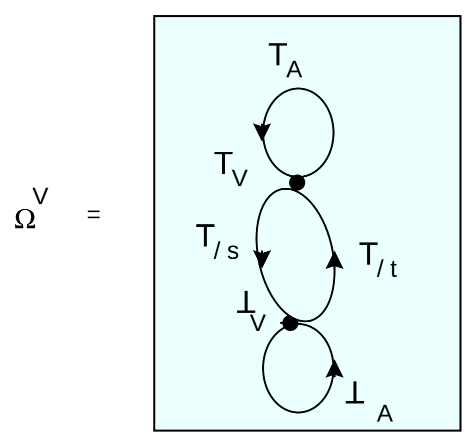
relations. For instance, let us calculate the source of $\top_A$

$$\begin{tikzcd}
 & A \arrow[r, dashed, "\top_A"] & \Omega^V \\
V \arrow[ur, "s"] \arrow[urr] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & A \times V \arrow[r, "\top_A"] & \Omega \\
V \times V \arrow[ur, "s \times 1_V"] \arrow[urr, "\top_A.(s \times 1_V)"'] & &
\end{tikzcd}$$



Graphically:

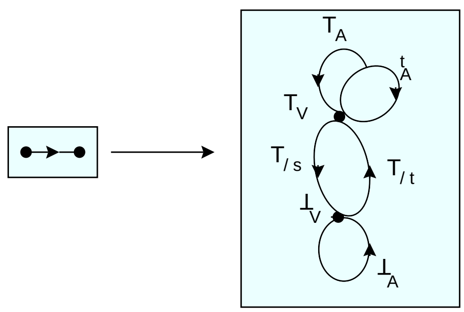


Comparing Ω with Ω^V we see that the loop t_L was lost in the process. We remark that there are only two external objects of Ω^V.

As a last example, let us begin the calculations for Ω^A :

$$\dfrac{\dfrac{\dfrac{V\longrightarrow\Omega^A}{V\times A\longrightarrow\Omega}}{2V\longrightarrow\Omega}}{\text{4 vertices}}$$

$$\dfrac{\dfrac{\dfrac{A\longrightarrow\Omega^A}{A\times A\longrightarrow\Omega}}{A+2V\longrightarrow\Omega}}{A\longrightarrow\Omega \qquad \dfrac{2V\longrightarrow\Omega}{\text{4 arrows}}}$$



5 arrows

20 arrows

EXERCISE 9.5.2

Calculate the incidences relations for these 20 arrows!

10 Doctrines

10.1 Propositional doctrines

In this chapter we study several ‘doctrines’ or algebraic counterparts to corresponding propositional logics. All of them will be enrichments of the doctrine of distributive lattices. This doctrine corresponds to ‘coherent propositional logic’. The interested reader may consult [34] for an exposition of this logic in terms of Gentzen rules of inference. The correspondence in question is realized in terms of the Lindembaum-Tarski algebra of a logic.

A *Heyting algebra* H is a bounded lattice such that the (order preserving map)

$$(\ ) \wedge a : H \longrightarrow H$$

has a right adjoint $a \to (\ )$, for each $a \in H$. Equivalently (by spelling this adjointness), a Heyting algebra is a bounded lattice with a binary operation $\to$ such that for every $x, y, z \in H$, we have the equivalence

$$\frac{z \leq x \to y}{z \wedge x \leq y}$$

We define a unary operation $\neg$ by $\neg a = a \to 0$. It is easy to show that a Heyting algebra is a distributive lattice. Indeed, since $(\ ) \wedge a$ has a right adjoint, it preserves (existing) sups. In particular, it preserves $b \vee c$. Thus, $(b \vee c) \wedge a = b \wedge a \vee c \wedge a$, i.e., distributivity holds.

Heyting algebras constitute a category by defining a *Heyting morphism* to be a lattice morphism which preserves $\to$.

A *co-Heyting algebra* K is a poset whose dual is a Heyting algebra. Equivalently, by noticing that $((\ ) \wedge b)^{op} = b \vee (\ )$, a co-Heyting algebra is a bounded lattice such that the order preserving map

$$b \vee (\ )$$

has a left adjoint $(\ )\backslash b$, for each $b \in K$. By spelling this condition once again, a co-Heyting algebra turns out to be a bounded lattice with a binary operation $\backslash$ such that for every $x, y, z \in K$, we have the following equivalences:

$$\frac{x \backslash y \leq z}{x \leq y \vee z}$$

Since distributivity is self-dual, a co-Heyting algebra is also a distributive lattice. We define a unary operation $\sim$ by $\sim x = 1 \backslash x$.

Similarly, co-Heyting algebras constitute a category by defining a *co-Heyting morphism* to be a lattice morphism which preserves $\backslash$.

To summarize:

Algebra	Heyting	co-Heyting
Adjunction	$(\,) \wedge x \dashv x \rightarrow (\,)$	$(\,) \backslash y \dashv y \vee (\,)$
Adjoint	$r(y) = x \rightarrow y$	$l(z) = z \backslash y$
Unit	$y \leq x \rightarrow (y \wedge x)$	$x \leq y \vee (x \backslash y)$
Counit	$(x \rightarrow z) \wedge x \leq z$	$y \vee (z \backslash y) \leq z$

We define a *bi-Heyting algebra* to be a Heyting algebra which is also a co-Heyting algebra. Once again, bi-Heyting algebras constitute a category under the obvious notion of morphism.

EXERCISE 10.1.1

(1) Show that every finite distributive lattice is a bi-Heyting algebra

(2) Let $\mathbb{A}$ be a co-Heyting algebra and let $x \in \mathbb{A}$. Define the *boundary* of x, ∂x, by $\partial x = x \wedge \sim x$. Show the following formulas:

$\partial(x \wedge y) = (y \wedge \partial x) \vee (x \wedge \partial y)$

$\partial(x \vee y) \vee \partial(x \wedge y) = \partial x \vee \partial y$

EXERCISE 10.1.2

(1) Let

$$P \underset{g}{\overset{f}{\leftrightarrows}} Q$$

be two functions between posets such that

$$\forall x \in P \; \forall y \in Q \; (f(y) \leq x \Leftrightarrow y \leq g(x))$$

Show that f, g are order preserving functions.

(2) Let

$$P \underset{g}{\overset{f}{\leftrightarrows}} Q$$

be order preserving maps between posets such that $f \dashv g$.

(i) Show that $fgf = f$ and $gfg = g$.

(ii) From (i) conclude that the following are equivalent

  (1) g is injective.

  (2) f is surjective.

  (3) $fg = 1_P$.

10.2 Predicate doctrines

This section is an introduction to predicate doctrines. These are categorical counterparts of first-order many-sorted logical theories. We shall briefly review the doctrines of coherent categories, Heyting categories, co-Heyting categories, bi-Heyting categories and L-categories. The first two correspond to many-sorted coherent and intuitionistic logic, respectively. The last two, to our knowledge, have not been formulated as 'logics' in the literature, although there are enough indications (in the literature) to do so. (See exercise 10.2.3) Thus, this section may be considered as an introduction to the basic notions of categorical logic. The interested reader may consult [34] as well as the references quoted in that paper.

A category $\mathbb{A}$ is *coherent* if

(i) It has $\mathbb{1}$ and pullbacks, i.e., finite limits

(ii) For every object X, the poset $P(X)$ of subobjects of X is a $\vee$-lattice (thus it is a lattice, because of (i))

(iii) For every $f : X \longrightarrow Y$ the pullback functor $f^* : P(Y) \longrightarrow P(X)$ is a lattice morphism

(iv) For every $f : X \longrightarrow Y$, the pullback functor $f^* : P(Y) \longrightarrow P(X)$ has a left adjoint (*existential quantifier*): $\exists_f \dashv f^*$.

(v) If

$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "u'"'] & U \arrow[d, "u"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

is a pullback, then the diagram

$$\begin{tikzcd} P(V) \arrow[r, "\exists_{f'}"] & P(U) \\ P(Y) \arrow[u, "(u')^*"] \arrow[r, "\exists_f"'] & P(X) \arrow[u, "u^*"'] \end{tikzcd}$$

is commutative.

A functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ between coherent categories is *coherent* iff it preserves finite limits, suprema of two sub-objects of an object and existential quantifiers $\exists_f$. In more detail, preservation of suprema may be expressed as follows: whenever X is an object of $\mathbb{A}$, $F(A_1 \vee A_2) = F(A_1) \vee F(A_2)$ for $A_1, A_2 \in P(X)$. Preservation of existential quantifier, on the other hand, may be expressed as $F(\exists_f(A)) = \exists_{F(f)}F(A)$, whenever $f : X \longrightarrow Y \in \mathbb{A}$ and $A \in P(X)$.

Examples of coherent categories

• The most basic example of coherent category is *Sets*. It is left as an exercise to show that for any $\mathbb{C}$, the category of $\mathbb{C}$-sets is coherent.

• Another, 'syntactical' examples is provided by the construction of a coherent category from a coherent theory , a construction that parallels the Lindembaum-Tarski construction of a Boolean algebra out of a classical propositional theory (see e.g. [29] or [34].)

A category is *Heyting* if it is coherent and

(vi) For every $f : X \longrightarrow Y$, the pullback functor $f^* : P(Y) \longrightarrow P(X)$ has a right adjoint $\forall_f$ (*universal quantifier*) as well as a left adjoint $\exists_f$.

A functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ between Heyting categories is *Heyting* if it is coherent and preserves the universal quantifiers $\forall_f$. This last condition may be described as follows: $F(\forall_f A) = \forall_{F(f)} F(A)$, whenever $f : X \longrightarrow Y \in \mathbb{A}$ and $A \in P(X)$.

Examples

• Once again, an example is given by the category of $\mathbb{C}$-sets. This follows from the preceding exercise and the description of quantifiers in that category (see section 9.3)

• A 'syntactical' example is provided by the previous construction applied now to a many-sorted intuitionistic theory.

A category is *bi-Heyting* iff it is Heyting and

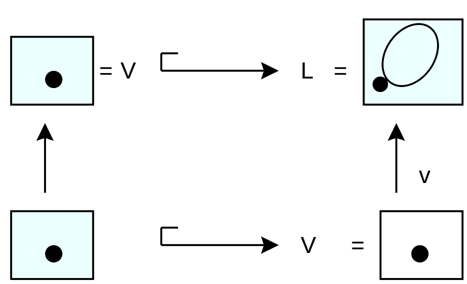
(vii) For every X, the lattice $P(X)$ is a co-Heyting algebra, i.e., $P(X)$ has a difference

A functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ between bi-Heyting categories is *bi-Heyting* if it is Heyting and preserves the difference operation, i.e., if $A_1, A_2 \in P(X)$, $F(A_1 \setminus_X A_2) = F(A_1) \setminus_{F(X)} F(A_2)$, for every object X of $\mathbb{A}$.

Examples

The existence of a difference in the category of $\mathbb{C}$-sets was described in detail in section 9.1, showing that this category is bi-Heyting.

Here is an example that shows that substitution does not commute with difference, in fact, not even with supplement: consider the pullback diagram



Then $v^*(\sim_L V) = v^*(L) = V$, but $\sim_V v^*(V) = \sim_V V = \emptyset$. In particular, this shows that substitution cannot be defined by recursion on formulas for bi-intuitionistic logic.

Similarly, Leibniz rule cannot be formulated in a simple minded way as the following shows: let $\Delta_F \hookrightarrow F \times F$ to be the diagonal in the category of reflexive graphs. A simple computation shows that $\partial\Delta_F \neq \emptyset$. On the other hand, an application of Leibniz rule , in its usual formulation, yields

$$(x = y) \wedge \sim (x = y) \vdash_{x,y} \sim (x = x)$$

i.e., $\partial\Delta_F = \emptyset$. The trouble is the same, although it appears in a slightly different form: the substitution $[x/y]$ in $\sim (x = y)$ is not $\sim (x = x)$!

`EXERCISE 10.2.1`

Compute $\partial\Delta_F$ in the category of reflexive graphs.

Remark 10.2.1 By applying the Beck-Chevalley condition to the pullback diagram (built canonically in $\mathbb{C}$-Sets)

$$\begin{tikzcd} Y \times_X X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "1_X"] \\ Y \arrow[r, "t"'] & X \end{tikzcd}$$

we conclude that

$$t^* A = \exists_{\pi_1} \pi_2^* A$$

whenever $A \in P(X)$.

Notice that in $Sets$, $\exists_{\pi_1} \pi_2^* A = \{y \in Y : \exists x \in X(x = t(y) \wedge x \in A)\}$. Thus, the above formula is the categorical counterpart of the Tarski formula for a proper substitution in classical logic: $\phi[t/x] \leftrightarrow \exists x(x = t \wedge \phi)$. This formula, however, has universal validity, regardless of the possibility of permuting pullbacks and logical operations.

`EXERCISE 10.2.2`

Define bi-intuitionistic logic as a formal system in such a way as to obtain an equivalence between this logical doctrine and its categorical counterpart. Hint: modify the usual Gentzen's formalization by adding a rule corresponding to the difference operation. What about substitution in this system?

A category is an *L-category* iff it is Heyting and

(viii) For every projection map $Y \times X \xrightarrow{\pi} X$ the pullback functor

$$P(X) \xrightarrow{\pi^*} P\,(Y \times X)$$

satisfies the following condition

$$\forall_\pi(\pi^*(A) \vee R) = A \vee \forall_\pi R$$

where $A \in P(X)$ and $R \in P(Y \times X)$.

`EXERCISE 10.2.3`

Define the L-logic as a formal system, following the suggestions of the preceeding exercise.

One of the recurrent themes of this book is the interaction between geometry and logic. The following shows the 'geometrical' content of the axiom defining an L-category at least for bi-Heyting categories:

Proposition 10.2.2 *Let $\mathbb{A}$ be a bi-Heyting category. Then the following conditions are equivalent:*

(1) For every $A \in P(X)$ and every $B \in P(Y)$

$$\partial(A \times B) = (\partial A \times B) \vee (A \times \partial B) \quad \textit{('tin can formula')}$$

(2) For every projection $\pi : Y \times X \longrightarrow X$ and every $A \in P(X)$

$$\pi^*(\sim_X A) = \sim_{Y \times X} \pi^*(A)$$

(3) For every projection $\pi : Y \times X \longrightarrow X$ and every couple $A_1, A_2 \in P(X)$

$$\pi^*(A_1 \setminus_X A_2) = \pi^*(A_1) \setminus_{Y \times X} \pi^*(A_2)$$

(4) For every projection $\pi : Y \times X \longrightarrow X$, every $A \in P(X)$ and every $R \in P(Y \times X)$

$$\forall_\pi(\pi^*(A) \vee R) = A \vee \forall_\pi(R).$$

Proof.
We first notice that if $\pi : Y \times X \longrightarrow X$ is a projection and $A \in P(X)$, then $\pi^*(A) = Y \times A$.

(1) $\rightarrow$ *(2)*: We shall prove that $Y \times \sim A \leq \sim (Y \times A)$, leaving the other direction as an easy exercise.

Assume that $\sim (Y \times A) \leq R$, i.e., $Y \times X = (Y \times A) \vee R$. By intersecting with $Y \times \sim A$, we obtain $Y \times \sim A \leq ((Y \times A) \wedge (Y \times \sim A)) \vee R$. Let us rewrite $(Y \times A) \wedge (Y \times \sim A) = Y \times \partial A$. From *(1)* (with $B = Y$), $Y \times \partial A = \partial(Y \times A)$. Thus, $Y \times \sim A \leq ((Y \times A) \wedge \sim (Y \times A)) \vee R \leq \sim (Y \times A)) \vee R = R$.

(2) $\rightarrow$ *(3)*: Let $\pi : Y \times A_1 \longrightarrow A_1$ be a projection. By (2),

$$\pi^*(\sim (A_1 \wedge A_2) = \sim Y \times (A_1 \wedge A_2).$$

On the other hand, the following formulas hold:

$$\sim_{A_1} (A_1 \wedge A_2) = A_1 \setminus_X A_2$$

$$\sim_{Y \times A_1} (Y \times (A_1 \wedge A_2)) = Y \times A_1 \setminus_X Y \times A_2$$

Let us check the first; the other is left as an exercise. We have the following equivalences

$$\begin{array}{c} \sim_{A_1} (A_1 \wedge A_2) \leq C \\ \hline \sim_{A_1} (A_1 \wedge A_2) \leq C' \\ \hline A_1 = (A_1 \wedge A_2) \vee C' \\ \hline A_1 = A_2 \vee C' \\ \hline A_1 \setminus_X A_2 \leq C' \\ \hline A_1 \setminus_X A_2 \leq C \end{array}$$

with $C' = C \wedge A_1$

(3) $\leftrightarrow$ *(4)*: This is a consequence of the following chain of equivalences:

$$\begin{array}{c} A' \leq \forall_\pi(\pi^*(A) \vee R) \\ \hline \pi^*(A') \leq \pi^*(A) \vee R \\ \hline \pi^*(A') \setminus \pi^*(A) \leq R \\ \hline A' \setminus A \leq \forall_\pi R \\ \hline A' \leq A \vee \forall_\pi R \end{array}$$

where $A' \in P(X)$.

(3) $\rightarrow$*(1)*: Let $A \in P(X)$ and $B \in P(Y)$. Applying *(3)* to $\pi : B \times X \longrightarrow X$ we have $B \times \sim_X A = \sim_{B\times X} (B \times A)$. Intersecting with $B \times A$, we obtain $(B \times A) \wedge (B \times \sim_X A) = (B \times A) \wedge \sim_{B\times X} (B \times A)$, i.e.,

$$B \wedge \partial A = (B \times A) \wedge (B \times X \setminus_{B\times X} B \times A)$$

Considering the projection $\pi : Y \times A \longrightarrow Y$ and proceeding as before, we obtain similarly

$$\partial B \times A = (B \times A) \wedge (Y \times A \setminus_{Y\times A} B \times A)$$

Therefore,

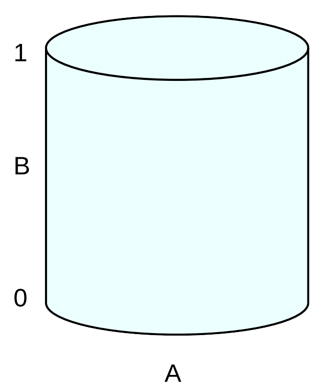
$$(\partial B \times A) \vee (B \times \partial A) = (B \times A) \wedge [(B \times X \setminus_{B\times X} B \times A) \vee (Y \times A \setminus_{Y\times A} B \times A)].$$

The following chain of equivalences

$$\begin{array}{c}
(B \times X \setminus_{B\times X} B \times A) \vee (Y \times A \setminus_{Y\times A} B \times A) \leq Z \\
\hline
(B \times X \setminus_{B\times X} B \times A) \leq Z \ , \ (Y \times A \setminus_{Y\times A} B \times A) \leq Z \\
\hline
B \times X \leq (B \times A) \vee Z \ , \ Y \times A \leq (B \times A) \vee Z \\
\hline
(B \times X) \vee (Y \times A) \leq (B \times A) \vee Z \\
\hline
Y \times X \leq (B \times A) \vee Z \\
\hline
\sim_{Y\times X} (B \times A) \leq Z
\end{array}$$

yields $(B \times X \setminus_{B\times X} B \times A) \vee (Y \times A \setminus_{Y\times A} B \times A) = \sim_{Y\times X} (B \times A)$, concluding the proof.

Remark 10.2.3 The name 'tin can formula' was coined by Lawvere to describe the boundary of a 'tin can' $A \times B$ as described in the figure



Notice that the boundary consists of two copies of A : the bottom $A \times \{0\}$ and the top $A \times \{1\}$ which altogether amount to

$$A \times \{0\} \vee A \times \{1\} = A \times \{0,1\} = A \times \partial B,$$

together with the lateral boundary of the can: $\partial A \times B$. Thus, the boundary of the tin can $A \times B$ may be expressed as

$$\partial(A \times B) = (\partial A \times B) \vee (A \times \partial B).$$

Example

A large class of examples is given by

Proposition 10.2.4 *Let $\mathbb{C}$ be a small category. Then the following conditions are equivalent:*

(1) For any morphism $A \xrightarrow{f} B \in \mathbb{C}$ there is a diagram in the category $\mathbb{C}$

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, hook, "g"'] & & B \arrow[dl, hook', "j"] \\
& C \arrow[ul, "i"] \arrow[ur, "h"'] &
\end{tikzcd}$$

such that $f = hg,\ hj = 1_B,\ ig = 1_A$

(2) The category $Sets^{\mathbb{C}^{op}}$ satisfies the tin can formula

Proof.
(1) $\to$ (2): Let $\pi : Y \times X \longrightarrow X$ be a projection and $F \in P(X)$. We shall prove

$$Y \times \sim F \subseteq \sim (Y \times F)$$

The other inequality is left as an exercise.

Let (y, x) be an A-figure of $Y \times \sim F$. Therefore (by definition of $\sim$) there is a change of figure $A \xrightarrow{f} B \in \mathbb{C}$ and a counterpart x' of x (i.e., a B-figure x' of X such that $x'f = x$) such that $x' \notin F$.

By applying the factorization to the morphism $A \xrightarrow{f} B$ we obtain the diagram

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, hook, shift right, "g"'] & & B \arrow[dl, hook', shift right, "j"'] \\
& C \arrow[ul, shift right, "i"'] \arrow[ur, shift right, "h"'] &
\end{tikzcd}
$$

with $hg = f, ig = 1_A, hj = 1_B$. Let $y' = yi, x'' = x'h$. These are counterparts of y and x, respectively: indeed

$$y'g = yig = y \quad \text{and} \quad x''g = x'hg = x'f = x.$$

Furthermore, $x'h \notin F$ (otherwise, $x' = x'hj \in F$, a contradiction). Thus, (y, x) is an A-figure of $\sim (Y \times F)$.

(2) $\rightarrow$ *(1)*: Assume that $A \xrightarrow{f} B \in \mathbb{C}$. Define X to be the sub $\mathbb{C}$-set of h_B generated by the morphisms $B' \xrightarrow{h'} B$ such that there is a diagram

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, hook, shift right, "g'"'] & & B \\
& C \arrow[ul, shift right, "i'"'] \arrow[ur, "h'"'] &
\end{tikzcd}
$$

with $f = h'g'$, $i'g' = 1_A$. Notice that f is an A-figure of X. Now, 1_B is either a B-figure of X or a B-figure of $\sim X$. In the first case, f has the required factorization.

In the second, $\sim X = h_B$. By the tin can formula (or rather the equivalent form given by proposition 10.2.2), $\sim (X \times h_A) = \sim X \times h_A = h_B \times h_A$. This implies that the A-figure $(f, 1_A)$ belongs to $\sim (X \times h_A)$. Therefore, by the formula that defines $\sim$, there is a change of figure $A \xrightarrow{g} C$ and 'counterparts' $C \xrightarrow{h} B$, $C \xrightarrow{i}$, of f and 1_A respectively, such that h is not a C-figure of X. Recall that the counterparts conditions, on the othe hand, say that $hg = f$ and $ig = 1_A$. I.e., they say precisely that h is a C-figure of X. Contradiction.

Remark 10.2.5 The above condition on factorization of morphisms was discovered by Lawvere who proved that *(1)* $\rightarrow$ *(2)*. The direction *(2)* $\rightarrow$ *(1)* on the other hand, was proved independently by H. Zolfaghari [44] and B. Loiseau (personal comunication).

The condition is satisfied, for instance, in the category $Sets^{\Delta^{op}}$ of *simplicial sets*, where Δ is the category of finite ordinals with non-decreasing

functions as morphisms. On the other hand, as we showed in theorem 9.2.6 the category $Sets^{\mathbb{C}^{op}}$ is Boolean iff $\mathbb{C}$ is a groupoid (i.e., all morphisms are isomorphisms).

WINDOW 10.2.1

Completeness theorems

To finish this section we shall say a few words about completeness theorems in their categorical guises for the logics mentioned in this section. Let $\mathbb{A}$ be a coherent category and $Mod(\mathbb{A})$ be the category of *models* whose objects are coherent functors from $\mathbb{A}$ into *Sets* and whose morphisms are natural transformations between such functors. Notice that we have a canonical *evaluation* functor $\mathbb{A} \xrightarrow{ev} Sets^{Mod(\mathbb{A})}$ defined on objects by $ev(A)(F) = F(A)$ and on morphisms $A \xrightarrow{f} B$ as the natural transformation $ev(A) \xrightarrow{ev(f)} ev(B)$ whose Fth component $ev(f)_F : ev(A)(F) \longrightarrow ev(B)(F)$ is simply $ev(f)_F = F(f)$.

The main result is the following

Theorem 10.2.6 (A. Joyal) *Let $\mathbb{A}$ be a coherent category. Then the evaluation functor*

$$\mathbb{A} \xrightarrow{ev} Sets^{Mod(\mathbb{A})}$$

is faithful, coherent and conditionally Heyting.

We say that a functor F between coherent categories is *conditionally Heyting* if whenever $\forall_f A$ exists for $f : X \longrightarrow Y$ in the domain category, with $A \in P(X)$, then $\forall_{F(f)} F(A)$ exists in the target category and

$$F(\forall_f(A)) = \forall_{F(f)} F(A).$$

Remark 10.2.7 Since the category $Mod(\mathbb{A})$ is not small in general, we cannot define the functor category $Sets^{Mod(\mathbb{A})}$. However, it is possible to replace the large category $Mod(\mathbb{A})$ by a *small* full subcategory $mod(\mathbb{A})$ in the above

theorem, provided that $\mathbb{A}$ is small. (This is a consequence of the Löwenheim-Skolem-Tarski theorem and the basic equivalence between coherent categories and coherent theories described in [34]) Thus, the above result is really a condensed formulation of the following

Theorem 10.2.8 *Let $\mathbb{A}$ be a small coherent category. Then there is a small full subcategory $mod(\mathbb{A})$ of $Mod(\mathbb{A})$ such that the evaluation functor*

$$\mathbb{A} \xrightarrow{ev} Sets^{mod(\mathbb{A})}$$

is faithful, coherent and conditionally Heyting.

Remark 10.2.9 Coherence and faithfulness constitute the categorical statement of completeness for coherent theories, which is equivalent to Gödel's completeness theorem.

A consequence of the above theorem is a Kripke-type completeness theorem for Heyting categories:

Corollary 10.2.10 *Let $\mathbb{A}$ be a small Heyting category. Then there is a small full subcategory $mod(\mathbb{A})$ of $Mod(\mathbb{A})$ such that the evaluation functor*

$$\mathbb{A} \xrightarrow{ev} Sets^{mod(\mathbb{A})}$$

is faithful and Heyting.

Remark 10.2.11 We should remark that in the Heyting and bi-Heyting doctrines, $Mod(\mathbb{A})$ remains the category of *coherent* functors from $\mathbb{A}$ into $Sets$, and *not*, as one could believe, the category of Heyting or bi-Heyting functors.

For bi-Heyting categories, the choice of the subcategory $mod(\mathbb{A})$ is more delicate: saturated models are required, but a slightly weaker form of the theorem remains true (see [35]):

Theorem 10.2.12 (Makkai-Reyes) *Let $\mathbb{A}$ be a small bi-Heyting category. Then there is a small full subcategory $mod(\mathbb{A})$ of $Mod(\mathbb{A})$ such that the evaluation functor*

$$\mathbb{A} \xrightarrow{ev} Sets^{mod(\mathbb{A})}$$

is faithful and bi-Heyting.

Remark 10.2.13 For L-categories, there is a completeness theorem that uses the notion of the topos of types of an L-category (defined in terms of Grothendieck topologies), rather than functor categories. Thus, contrary to the other completeness theorems, even the notions used in the formulation of the theorem go beyond the scope of this book. (See [35]).

Remark 10.2.14 Similarly, there are completeness theorems for propositional logics in their categorical guises. See for this results and further developments [6] and [7]

11 Geometric morphisms

Let $\mathbb{A}$ and $\mathbb{B}$ be two categories with pullbacks and terminal objects, a *geometric morphism*

$$\mathbb{A} \xrightarrow{u} \mathbb{B}$$

is a pair of adjoint functors $u = (u^*, u_*)$, $u^* \dashv u_*$

$$\mathbb{A} \underset{u_*}{\overset{u^*}{\rightleftarrows}} \mathbb{B},$$

such that u^* preserves the pullbacks and the terminal object (so, in fact, preserves finite limits). The functor u^* is called the *inverse image functor*, whereas u_* is called the *direct image functor*. The direction of the geometric morphism is given by u_* (the *geometric sense*). We remark that u^* preserves the colimits since it has a right adjoint and that u_* preserves the limits, and not only the finite ones, since it has a left adjoint.

11.1 The pair (Δ, Γ)

An example of a geometric morphism is the pair (Δ, Γ) :

$$Sets^{\mathbb{C}^{op}} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} Sets$$

defined as follows. If S is a set, ΔS is the $\mathbb{C}$-set whose F-figures and incidence relations are

$$\frac{F \dashrightarrow \Delta S}{s \in S} \qquad \begin{tikzcd} & F \arrow[r, dashed, "s"] & \Delta S \\ F' \arrow[ur, "f"] \arrow[urr, dashed, "s.f=s"'] & & \end{tikzcd}$$

On a function $S \xrightarrow{\phi} T$, $\Delta\phi : \Delta S \longrightarrow \Delta T$ is defined to be the natural transformation whose F-component $\Delta\phi_F$ evaluated at s is $\Delta\phi_F(s) = \phi(s)$. We call Δ the *constant* or *discrete* functor.

Let us prove that Δ preserves pullbacks. Assume that

$$\begin{tikzcd} S_1 \arrow[r] \arrow[d] & S_2 \arrow[d] \\ S_3 \arrow[r] & S_4 \end{tikzcd}$$

is a pull back in $Sets$. We have to show that

$$\begin{tikzcd} \Delta S_1 \arrow[r] \arrow[d] & \Delta S_2 \arrow[d] \\ \Delta S_3 \arrow[r] & \Delta S_4 \end{tikzcd}$$

is a pullback in $Sets^{\mathbb{C}^{op}}$. By definition of pullbacks in this category, this is the same as showing that for all F the following diagram is a pullback in $Sets$:

$$\begin{tikzcd} \Delta S_1(F) \arrow[r] \arrow[d] & \Delta S_2(F) \arrow[d] \\ \Delta S_3(F) \arrow[r] & \Delta S_4(F) \end{tikzcd}$$

But this is just the original diagram (by definition of Δ).

It is easy to show that Δ preserves the terminal object ($\Delta 1 = \mathbb{1}$).

We must now define Γ such that $\Delta \dashv \Gamma$. If Γ exists, this adjunction implies the existence of a bijection

$$\frac{\Delta S \longrightarrow X \quad \in Sets^{\mathbb{C}^{op}}}{S \longrightarrow \Gamma X \quad \in Sets}$$

In particular, for $S = 1$,

$$\frac{\mathbb{1} \xrightarrow{\xi} X \qquad \in Sets^{\mathbb{C}^{op}}}{\xi \in \Gamma X \qquad \in Sets}$$

Thus, the definition of Γ is forced on objects: $\Gamma X = \mathbb{C}(\mathbb{1}, X)$. But more is true: its definition is forced on morphisms, since in the above bijection commutative triangles should correspond to commutative triangles (see section 8). In particular we have the bijection

$$\frac{\begin{tikzcd} \Delta 1 \arrow[r, "\xi"] \arrow[drr] & X \arrow[dr, "\phi"] & \\ & & X' \end{tikzcd}}{\begin{tikzcd} 1 \arrow[r, "\xi"] \arrow[drr] & \Gamma X \arrow[dr, "\Gamma\phi"] & \\ & & \Gamma X' \end{tikzcd}} \qquad \Gamma\phi(\xi) = \phi \circ \xi$$

Thus Γ is forced to be the functor $\mathbb{C}(\mathbb{1}, -)$. We call Γ the *points* or *global section* functor.

We must verify that $\Delta \dashv \Gamma$. We define the unit of the adjunction to be the natural transformation

$$Id_{Sets} \xrightarrow{\eta} \Gamma\Delta$$

whose S-component evaluated at s is $\eta_S(s) = \Delta s$, where we have identified $s \in S$ with $s : 1 \longrightarrow S$.

Similarly, the counit of the adjunction is the natural transformation

$$\Delta\Gamma \xrightarrow{\epsilon} Id_{Sets^{\mathbb{C}^{op}}}$$

whose X-component $\Delta\Gamma X \xrightarrow{\epsilon_X} X$ is defined to be $(\epsilon_X)_F(\xi) = \xi_F$, where $F \in \mathbb{C}$, $X \in Sets^{\mathbb{C}^{op}}$ and $\mathbb{1} \xrightarrow{\xi} X$ is an element of ΓX.

To finish the proof that $\Delta \dashv \Gamma$, it is enough (by proposition 8.1.3) to show that the following identities are verified: $\epsilon_{\Delta S} \circ F(\eta_S) = 1_{\Delta S}$ and $\Gamma(\epsilon_X) \circ \eta_{\Gamma X} = 1_{\Gamma X}$ for $S \in Sets$ and $X \in Sets^{\mathbb{C}^{op}}$. This task is left to the reader.

We prove now that *Sets* is a sort of terminal object in the category whose objects are $\mathbb{C}$-sets and whose morphisms are geometric morphisms between them.

Proposition 11.1.1 *The pair* (Δ, Γ) *is essentially the only geometric morphism from* $Sets^{\mathbb{C}^{op}}$ *into* $Sets$.

Proof. Indeed, let us suppose that (p^*, p_*) is any geometric morphism. Let S be a set. Then $S = \bigsqcup_{s\in S}\{s\}$. But since p^* preserves colimits,

$$p^*S = p^* \bigsqcup_{s\in S}\{s\} \simeq p^* \bigsqcup_{s\in S} 1.$$

Furthermore p^* preserves the terminal object. So,

$$\bigsqcup_{s\in S} p^*1 = \bigsqcup_{s\in S} \mathbb{1} = \Delta S$$

Thus $p^* \simeq \Delta$. Similarly $p_* \simeq \Gamma$ since adjoints are unique (up to isomorphism).

To finish this section we will study conditions under which the unit and the counit of the adjunction $\Delta \dashv \Gamma$ are isomorphisms.

Proposition 11.1.2 *The unit* $Id \xrightarrow{\eta} \Gamma\Delta$ *is an isomorphism iff the functor* Δ *is full and faithful*

Proof.
We shall prove that Δ is full, provided that the unit η is an isomorphism. The rest of the proof is left to the reader.

Let $\Phi : \Delta X \longrightarrow \Delta Y$ be given. Define $f = \eta_Y{}^{-1}\Gamma(\Phi)\eta_X$ as described in the diagram

$$\begin{tikzcd}
\Gamma\Delta X \arrow[d, "\Gamma(\Phi)"'] & X \arrow[l, "\sim", "\eta_X"'] \arrow[d, "\eta_Y^{-1}\Gamma(\Phi)\eta_X = f"] \\
\Gamma\Delta Y & Y \arrow[l, "\sim", "\eta_Y"']
\end{tikzcd}$$

We claim that $\Phi = \Delta(f)$. Indeed the following two diagrams commute (the first because of the naturality of η and the second by the definition of f.)

$$\begin{tikzcd}
X \arrow[r, "\sim", "\eta_X"'] \arrow[d, "f"'] & \Gamma\Delta X \arrow[d, "\Gamma\Delta(f)"] \\
Y \arrow[r, "\sim", "\eta_Y"'] & \Gamma\Delta Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\sim", "\eta_X"'] \arrow[d, "f"'] & \Gamma\Delta X \arrow[d, "\Gamma(\Phi)"] \\
Y \arrow[r, "\sim", "\eta_Y"'] & \Gamma\Delta Y
\end{tikzcd}$$

Therefore $\Gamma(\Phi) = \Gamma\Delta(f)$ and so $\Delta\Gamma(\Phi) = \Delta\Gamma\Delta(f)$. A glance at the diagram

$$\begin{tikzcd}
\Delta X \arrow[r, "\sim", "\Delta\eta_X"'] \arrow[d, shift right, "\Phi"'] \arrow[d, shift left, "\Delta f"] & \Delta\Gamma\Delta X \arrow[d, shift right, "\Delta\Gamma\Phi"'] \arrow[d, shift left, "\Delta\Gamma\Delta f"] \arrow[d, phantom, "="] \\
\Delta Y \arrow[r, "\sim", "\Delta\eta_X"'] & \Delta\Gamma\Delta Y
\end{tikzcd}$$

concludes the proof of fullness.

This proposition is a particular case of the following result that may be found in [28]:

Proposition 11.1.3 *Let $F \dashv G$ be an adjunction from $\mathbb{B}$ to $\mathbb{A}$. Then the counit $FG \xrightarrow{\epsilon} Id$ is an isomorphism iff G is full and faithful.*

Proposition 11.1.4 *Assume that $\mathbb{C}$ has at least one object. Then the following conditions are equivalent:*

(1) the counit $\Delta\Gamma \xrightarrow{\epsilon} Id$ is an isomorphism

(2) the functor Γ is full and faithful

(3) Δ is an equivalence of categories

(4) $\mathbb{C} \simeq \mathbb{1}$.

Proof.
Notice that *(1)* and *(2)* are equivalent by the previous proposition.

(4) $\rightarrow$ *(3)* and *(3)* $\rightarrow$ *(2)* are obvious

(1)$\rightarrow$ *(4)*: Take any $A \in \mathbb{C}$. By hypothesis,

$$\Delta\Gamma h_A \xrightarrow{\epsilon_A} h_A$$

is an isomorphism. Let $f,\ g : A \longrightarrow A$. By naturality of ϵ the diagram

$$\begin{tikzcd}
\Gamma h_A \arrow[r, "\sim", "(\epsilon_A)_A"'] & h_A(A) \\
\Gamma h_A \arrow[u, "1_{\Gamma h_A}"] \arrow[r, "\sim", "(\epsilon_A)_A"'] & h_A(A) \arrow[u, shift left, "-\circ f"'] \arrow[u, shift right, "-\circ g", shift left=3]
\end{tikzcd}$$

is commutative, i.e., $(\ )\circ f = (\ )\circ g$. By applying these maps to 1_A we conclude that $f = g$ and so $h_A(A) = \{1_A\}$. Thus $\Gamma(h_A)$ has one element.

Let B be an arbitrary object in $\mathbb{C}$. Then

$$\Gamma(h_A) \xrightarrow{(\epsilon_A)_B} h_A(B)$$

is an isomorphism. In particular, there is a unique map $B \longrightarrow A$ and this shows that every object in $\mathbb{C}$ is terminal, i.e., $\mathbb{C} \simeq \mathbb{1}$.

EXERCISE 11.1.1

Show that Δ is faithful iff $\mathbb{C}$ has at least one object.

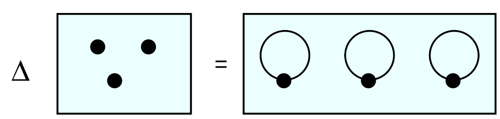
Examples of the pair (Δ, Γ) *for particular* $\mathbb{C}$*-Sets*

• Sets. In that case, $\Delta = \Gamma = Id$

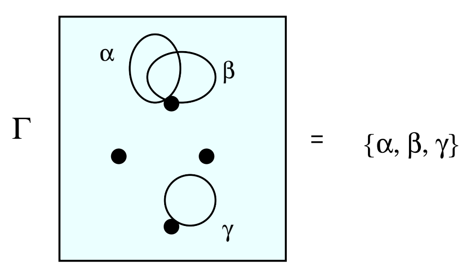
• Bouquets. Let S be a set containing three elements. Since Δ preserves the colimits and the terminal object,

$$\Delta(1+1+1) = \Delta(1) + \Delta(1) + \Delta(1) = \mathbb{1} + \mathbb{1} + \mathbb{1}$$

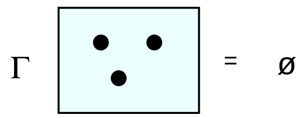
where $\mathbb{1}$ is the terminal object of the bouquets. This means that we obtain three loops:



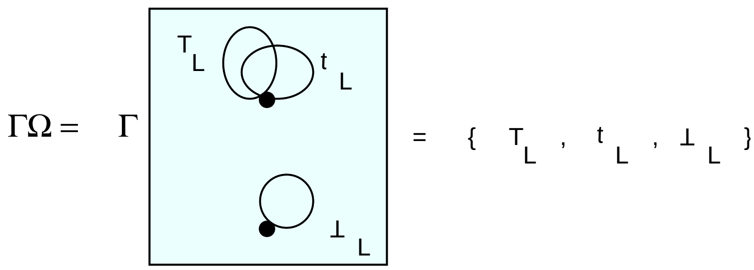
If X is a bouquet, ΓX is the set containing as elements the loops of the bouquet. Let us visualise this in three different bouquets:
1)



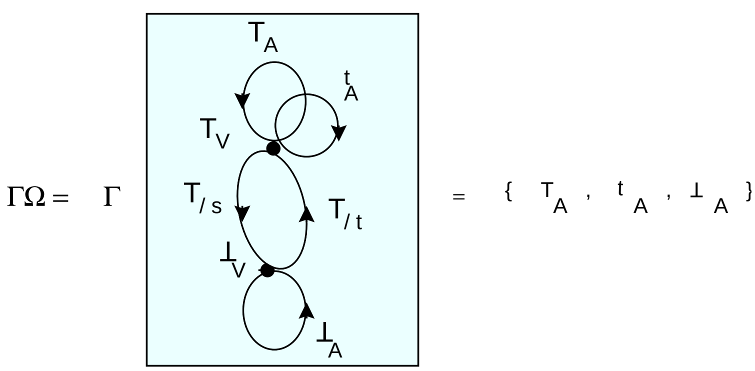
2)



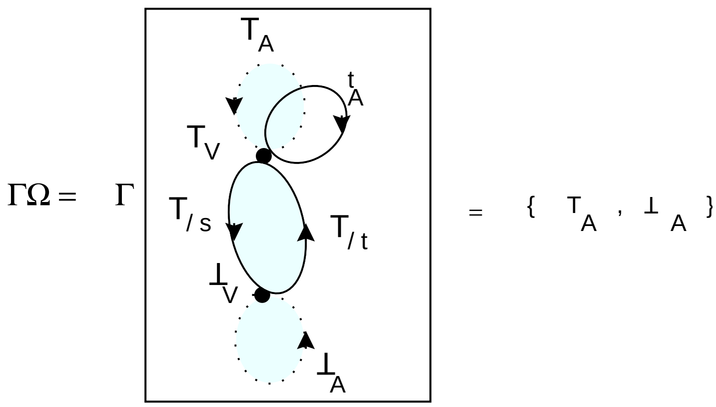
3)



• Graphs. Let S be a set, ΔS is the graph containing as many oriented loops (arrows whose source and target coincide) as there are elements in S. If X is a graph, ΓX is the set containing as elements the oriented loops of the graph. For instance,



• Rgraphs. Let S be a set, ΔS is the reflexive graph containing as many distinguished oriented loops as there are elements in S. If X is a reflexive graph, ΓX is the set containing as elements the distinguished oriented loops of the reflexive graph. For instance,



• Esets. Let S be a set, ΔS is the evolutive set containing as many oriented loops as there are elements in S. If X is an evolutive set, ΓX is the set containing the oriented loops of the evolutive set. In the case that $X = \Omega$, $\Gamma\Omega = \{0, 1\}$.

11.2 Sieves and $\Gamma\Omega$

From the defintion of $\Gamma\Omega$ in the general case, we obtain the equivalences

$$\frac{\dfrac{\sigma \in \Gamma\Omega}{\mathbb{1} \overset{\sigma}{\hookrightarrow} \Omega}}{Z \hookrightarrow \mathbb{1}} \quad (\Omega \text{ classifies sub-objects})$$

The presheaf $Z \hookrightarrow \mathbb{1}$ is determined by the set $K_Z = \{F | Z(F) = 1\}$. We remark that K_Z has the property that if $F \in K_Z$ and $F' \longrightarrow F \in \mathbb{C}$ then $F' \in K_Z$. In fact $Z(F') = 1$, since $Z(F) = 1$.

We say that $K \subseteq Obj(\mathbb{C})$ is a *sieve of* $\mathbb{C}$ iff $F \in K$ and $F' \xrightarrow{f} F \in \mathbb{C}$ implies that $F' \in K$.

From this discussion we may conclude

Proposition 11.2.1 *The set* $\Gamma\Omega$ *consists of the sieves of* $\mathbb{C}$ *.*

Examples of sieves of $\mathbb{C}$

• Graphs.

$$\Gamma\Omega = \{\emptyset, \{V\}, \{V, A\}\} = \{\perp_A, t_A, \top_A\}$$

- Rgraphs.

$$\Gamma\Omega = \{\emptyset, \{V, A\}\} = \{\perp_A, \top_A\}.$$

- Esets.

$$\Gamma\Omega = \{\emptyset, \{*\}\} = \{0, 1\}.$$

We recall the equivalences

$$\frac{\frac{F \overset{\sigma}{\dashrightarrow} \Omega}{\sigma \hookrightarrow h_F}}{(\sigma(F') \subseteq \mathbb{C}(F', F))_{F'} \text{ such that } (*)}$$

where

$$(*): \; f \in \sigma(F') \text{ and } F'' \xrightarrow{g} F' \in \mathbb{C} \Rightarrow f \circ g \in \sigma(F'')$$

If $\mathbb{C}$ has a terminal object, then a sieve is the same as a $\mathbb{1}$-figure of Ω :

$$\mathbb{1} \dashrightarrow \Omega$$

Let $\mathbb{C}$ be a category and let $Sets^{\mathbb{C}^{op}}$ be the category of $\mathbb{C}$-sets. In sections 9, 9.2 we defined truth functions as maps $\Omega^n \xrightarrow{\phi} \Omega$. In the sequel we shall limit ourselves to $n = 1$.

Proposition 11.2.2 *There is a natural bijection between the set of F-figures of Ω and the set of sieves of the comma category $\mathbb{C}/F$.*

Proof. Recall [28] the definition of the comma category $\mathbb{C}/F$: its objects are $\mathbb{C}$-morphisms $F' \xrightarrow{f} F$ and its morphisms $f \xrightarrow{\phi} g$ are commutative triangles

$$\begin{tikzcd} F' \arrow[rr, "\phi"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & F & \end{tikzcd}$$

On the other hand, an F-figure of Ω was defined to be a subpresheaf $\sigma \hookrightarrow h_F$. Let $K = \bigcup_{F'} \sigma(F')$. Then K is a sieve of $\mathbb{C}/F$. In the other direction, if K is such a sieve, define $\sigma(F') = K \cup \mathbb{C}(F', F)$. It is easy to show that these operations are in one-to-one correspondance.

We should expect that in the passage from the truth function $\Omega \xrightarrow{\phi} \Omega$ to $\Gamma\Omega \xrightarrow{\Gamma\phi} \Gamma\Omega$ some information is lost. In fact the loss may be spectacular. For instance, in *Esets* there are $\aleph_0$ truth functions $\Omega \xrightarrow{\phi} \Omega$ but there are only four functions $\Gamma\Omega = \{0,1\} \xrightarrow{\Gamma\phi} \Gamma\Omega = \{0,1\}$. We have, however, the following

Proposition 11.2.3 (A. Galli and M. Sagastume) *If* $\mathbb{C} = \mathbb{P}$*, a pre-ordered set, then there is a bijection*

$$\frac{\Omega \xrightarrow{\phi} \Omega}{\Phi : \Gamma\Omega \longrightarrow \Gamma\Omega \;\; \textit{such that (*)}}$$

where (*) is the condition: $p \in \Phi(K)$ *iff* $p \in \Phi(K \cap \downarrow p)$

Proof (sketch).
($\downarrow$): Apply Γ
($\uparrow$):

$$\begin{tikzcd} \Gamma\Omega \arrow[r, shift left, "- \cap \downarrow(p)"] & \Omega(p) \arrow[l, hook', shift left] \end{tikzcd}$$

where $\Gamma\Omega$ is the set of downward closed subsets of $\mathbb{P}$.

11.3 Essential geometric morphisms

In several examples, including all those studied up to now, the geometric morphisms are essential in the sense of the following definition.

A geometric morphism $\mathbb{A} \xrightarrow{p} \mathbb{B}$ is *essential* iff u^* has a left adjoint

$$u_! \dashv u^* \dashv u_*$$

We remark that if we have $u_! \dashv u^* \dashv u_*$, then automatically (u^*, u_*) is a geometric morphism.

Proposition 11.3.1 *The morphism* (Δ, Γ) *is essential, namely, there exists a functor* Π

$$\begin{tikzcd} Sets^{\mathbb{C}^{op}} \arrow[r, shift left=2, "\Pi"] \arrow[r, shift right=2, "\Gamma"'] & Sets \arrow[l, "\Delta"'] \end{tikzcd}$$

such that

$$\Pi \dashv \Delta \dashv \Gamma.$$

Proof.
This is a consequence of the definition of colimit and limit of a functor (see section 4.2). In fact, $\Pi = colim$ and $\Gamma = lim$. Here we shall give an alternative proof that provides a 'geometric' interpretation of the functor Π.

We first note that if Π exists, it must preserve the colimits, namely the glueings. Let us suppose that Π exists and let us calculate it in the case of graphs. The idea is the following, since every graph comes with a blueprint that gives instructions on how to glue copies of the generic figures, we calculate Π of the generic figures and then we use the blueprint to glue these Π 's. Let S be a set, for the vertices:

$$\frac{\dfrac{\dfrac{\Pi V \longrightarrow S}{V \longrightarrow \Delta S}}{S}}{1 \longrightarrow S}$$

From these equivalences and with the help of the following simple lemma, we conclude that $\Pi V = 1$.

Lemma 11.3.2 *Let $\mathbb{C}$ be a category. Assume that there is a natural bijection*

$$\frac{A \longrightarrow B}{A' \longrightarrow B}$$

Then $A \simeq A'$. Dually, assume that there is a natural bijection

$$\frac{A \longrightarrow B}{A \longrightarrow B'}$$

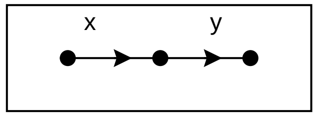
Then $B \simeq B'$.

For the arrows, we obtain

$$\frac{\dfrac{\dfrac{\Pi A \longrightarrow S}{A \xrightarrow{\sigma} \Delta S}}{\sigma \in S}}{1 \xrightarrow{\sigma} S}$$

Thus $\Pi A = 1$ (by the lemma).

Let us compute Π of the following graph:

G = 

By applying Π to its canonical blueprint

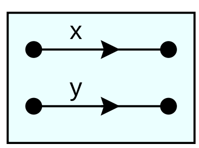
$$\begin{tikzcd} & A \arrow[dr, "x"] & \\ V \arrow[ur, "t"] \arrow[dr, "s"'] & & G \\ & A \arrow[ur, "y"'] & \end{tikzcd}$$

we obtain the colimit

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 1 \arrow[ur] \arrow[dr] & & \Pi G \\ & 1 \arrow[ur] & \end{tikzcd}$$

and hence, $\Pi G = 1$.

Let us compute Π of still another graph:

G = 

By applying Π to its canonical blueprint

$$\begin{tikzcd} A \arrow[dr, "x"] & \\ & G \\ A \arrow[ur, "y"'] & \end{tikzcd}$$

we obtain the colimit

$$
\begin{tikzcd}
1 \arrow[dr] & \\
& \Pi G \\
1 \arrow[ur] &
\end{tikzcd}
$$

and hence, $\Pi G = 2$.

These examples lead us to conjecture that ΠG is the set of connected components of G. We will show that this is so, not only for graphs but in the general case.

As we have done on other occasions we use the analytic method: given the properties of Π and postulating its existence, we conclude that Π is uniquely determined.
(1) If F is a generic figure then $\Pi h_F = 1$. In fact,

$$
\begin{array}{c}
\Pi h_F \longrightarrow S \\
\hline
\Delta S(F) \\
\hline
S \\
\hline
1 \longrightarrow S
\end{array}
$$

hence, $\Pi h_F = 1$
(2) Let us recall that every $\mathbb{C}$-set is the colimit of generic figures. In a more precise way, $X = colim\ U_X$, where

$$Fig(X) \xrightarrow{U_X} Sets^{\mathbb{C}^{op}}$$

is given by

$$(F \overset{\sigma}{\dashrightarrow} X) \quad \longmapsto \quad h_F$$

Since $\Pi \dashv \Delta$, Π has a right adjoint and hence it must preserve colimits. Therefore, the definition of Π is forced: $\Pi X = colim \Pi \circ U_X$. By (1), on the other hand, $\Pi \circ U_X = \delta 1$, where $\delta 1 : Fig(X) \longrightarrow Sets$ is the functor whose value is constantly 1. Therefore we find the final expression

$$\Pi X = colim \delta 1$$

When is the unit (respectively the counit) of the adjunction $\Pi \dashv \Delta$ an isomorphism? The answer is given by

Proposition 11.3.3 *(1) The counit* $\Pi\Delta \xrightarrow{\epsilon} Id$ *is an isomorphism iff* $\Pi\mathbb{1} = 1$.

(2) Assume that $\mathbb{C}$ *has at least one object. Then the unit* $Id \xrightarrow{\eta} \Delta\Pi$ *is an isomorphism iff* $\mathbb{C} \simeq \mathbb{1}$.

Proof.

(1) : Assume that $\Pi\mathbb{1} = 1$. The computation $\Pi\Delta S = \Pi \bigsqcup_S \mathbb{1} = \bigsqcup_S \Pi\mathbb{1} = \bigsqcup_S 1 = S$ concludes the non-trivial direction of the proof.

(2) : Assume that the unit is an isomorphism. In particular $h_A \xrightarrow{\epsilon_A} \Delta\Pi h_A$ is a bijection. But $\Pi h_A = 1$. Thus, $h_A \simeq \mathbb{1}$ and so every object in $\mathbb{C}$ is terminal, i.e., $\mathbb{C} \simeq \mathbb{1}$. The other direction is obvious.

Corollary 11.3.4 *The following conditions are equivalent*

(1) The counit $\Pi\Delta \xrightarrow{\epsilon} Id$ *is an isomorphism*

(2) The functor Δ *is full and faithful*

(3) $\Pi\mathbb{1} = 1$

11.4 Geometric morphisms in examples

Let us compute a few geometric morphisms between some of our favorite examples. We start with one between *Bouquets* and *Graphs*.

Notice that we have a forgetful functor

$$Bouquets \xrightarrow{U} Graphs$$

which considers a bouquet as a very special graph. In fact, we could say that U 'expresses' the fact that a bouquet *is* a graph with further properties. We shall prove that U has both left La and right Ra adjoints that we shall compute explicitly:

$$\begin{tikzcd} Graphs \arrow[r, shift left=2, "La"] \arrow[r, shift right=2, "Ra"'] & Bouquets \arrow[l, "U"'] \end{tikzcd}$$

with $La \dashv U \dashv Ra$.

We remark first that U applied to a vertex of a bouquet gives a vertex of the corresponding graph and U applied to a loop of a bouquet gives an arrow whose source and target coincide. Let G be a graph. We calculate now the vertices, the loops and the incidence relations of RaG.

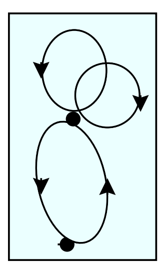
$$\begin{array}{c} V \longrightarrow RaG \quad \in \mathit{Bouquets} \\ \hline UV \longrightarrow G \quad \in \mathit{Graphs} \\ \hline V \longrightarrow G \quad \in \mathit{Graphs} \\ \hline \text{vertices of } G \end{array} \qquad \begin{array}{c} L \longrightarrow RaG \quad \in \mathit{Bouquets} \\ \hline UL \longrightarrow G \quad \in \mathit{Graphs} \\ \hline \mathbb{1} \longrightarrow G \quad \in \mathit{Graphs} \\ \hline \text{loops of } G \end{array}$$

and

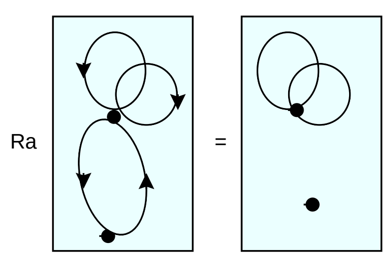
$$\begin{tikzcd} & L \arrow[r] & RaG \\ V \arrow[ur, "v"] \arrow[urr] & & \end{tikzcd}$$

$$\begin{tikzcd} & \mathbb{1} = UL \arrow[r] & G \\ V = UV \arrow[ur, "\exists!"] \arrow[urr] & & \end{tikzcd}$$

(We have used the same notation for the vertices of the *Bouquets* and for the vertices of the *Graphs*, it is clear from the context what we mean when we write $V = UV$.) So RaG is a bouquet that has the same vertices as G but contains only the loops of G. For instance, if G is the following graph



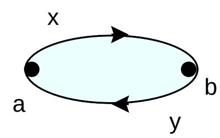
then



Similarly, we calculate the vertices and the loops of LaG.

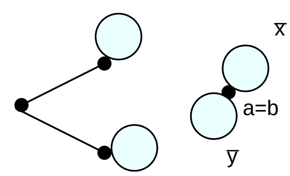
$$
\begin{array}{c}
LaV \longrightarrow X \quad \in Bouquets \\
\hline
V \longrightarrow UX \quad \in Graphs \\
\hline
V \longrightarrow X \quad \in Bouquets \\
\text{hence } LaV = V
\end{array}
\qquad
\begin{array}{c}
LaA \longrightarrow X \quad \in Bouquets \\
\hline
A \longrightarrow UX \quad \in Graphs \\
\hline
L \longrightarrow X \quad \in Bouquets \\
\text{hence } LaA = L
\end{array}
$$

Let us give an example, let G be the following graph

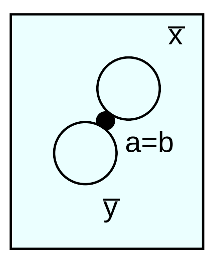


Let us look at the blueprint of G, and then at the action of La on the blueprint

$$
\begin{tikzcd}
 & A \arrow[dr, dashed, "x"] & \\
V \arrow[ur, shift left, "s"] \arrow[ur, shift right, "t"'] \arrow[dr, shift left, "s"] \arrow[dr, shift right, "t"'] & & G \\
 & A \arrow[ur, dashed, "y"'] &
\end{tikzcd}
\quad \overset{La}{\longmapsto} \quad
\begin{tikzcd}
 & L \arrow[dr, "\overline{x}"] & \\
V \arrow[ur, "v"] \arrow[dr, "v"'] & & LaG \\
 & L \arrow[ur, "\overline{y}"'] &
\end{tikzcd}
$$



And we conclude that

LaG=


Let us turn to another example, the geometric morphisms between *Esets* and *Graphs*. Once again, we may view an evolutive set as a special graph, a fact that we objectivize by means of a forgetful functor

$$Esets \xrightarrow{U} Graphs.$$

We will prove that U has both a left (La) and a right (Ra) adjoint, that we compute now:

$$Graphs \underset{Ra}{\overset{La}{\underset{U}{\rightleftarrows}}} Esets$$

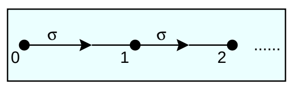
Let (X, f) be an evolutive set. Then $U(X, f)$ is the graph whose vertices are the elements of X and whose arrows are the pairs $(x, f(x))$ with $x \in X$. So we have

$$X \overset{(1_X, f)}{\longrightarrow} X \times X$$

$$x \longmapsto (x, f(x))$$

such that$(x, f(x)).s = x$ and $(x, f(x)).t = f(x)$.

We calculate now Ra and La. Let I be the generic chain

I = 

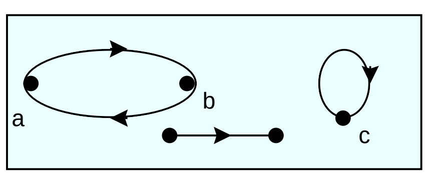
$$\begin{array}{c} I \longrightarrow RaG \qquad \in Esets \\ \hline UI \longrightarrow G \qquad \in Graphs \\ \hline \text{the chains in } G \end{array} \qquad \begin{array}{c} \begin{tikzcd} & I \arrow[r, "x"] & RaG \\ I \arrow[ur, "\sigma"] \arrow[urr] & & \end{tikzcd} \\ \hline \begin{tikzcd} & UI \arrow[r, "\overline{x}=x_0,x_1,\ldots"] & G \\ UI \arrow[ur, "U\sigma"] \arrow[urr, "\overline{x}\sigma=x_1,x_2,\ldots"'] & & \end{tikzcd} \end{array}$$

and

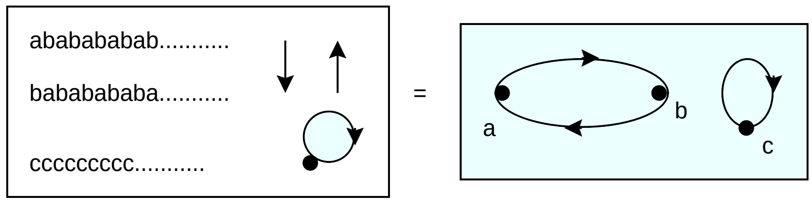
$$\begin{array}{c} LaV \longrightarrow (X, f) \qquad \in Esets \\ \hline V \longrightarrow U(X, f) \qquad \in Graphs \\ \hline I \longrightarrow (X, f) \qquad \in Esets \end{array}$$

Thus $LaV = I$. In the same way we find $LaA = I$, $La(s) = 1_I$ and $La(t) = \sigma$. We remark that having a vertex is the same as having a chain since we know exactly where a vertex goes by the evolution. The system is deterministic.

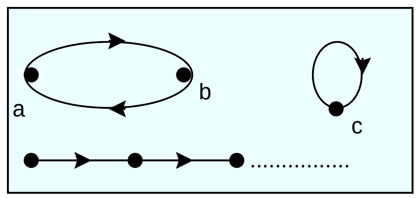
Let us see an example. Let G be the following graph



RaG =

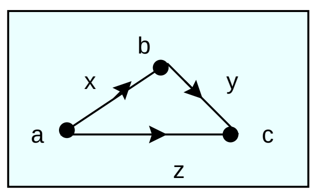


LaG =



In words, we can say that Ra extracts from G all the possible chains and that La adds to G all the necessary chains.

Let us look at another example. Let G be the following graph

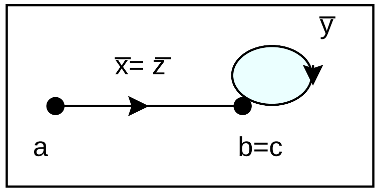


From G it is impossible to extract any chain. So $RaG = \emptyset$. We now study the blueprint of G and the action of La on it

$$
\begin{tikzcd}
 & A \arrow[ddr, "x"] & & & & I \arrow[ddr, "\overline{x}"] & \\
V \arrow[ur, "t"] \arrow[dr, "s"] & & & & I \arrow[ur, "\sigma"] \arrow[dr, "1"] & & \\
V \arrow[uur, "s"] \arrow[ddr, "s"] & A \arrow[r, "y"] & G & \overset{La}{\mapsto} & I \arrow[uur, "1"] \arrow[ddr, "1"] & I \arrow[r, "\overline{y}"] & LaG \\
V \arrow[ur, "t"] \arrow[dr, "t"] & & & & I \arrow[ur, "\sigma"] \arrow[dr, "\sigma"] & & \\
 & A \arrow[uur, "z"] & & & & I \arrow[uur, "\overline{z}"] &
\end{tikzcd}
$$

From that we can conclude that $\overline{x} = \overline{z}$, $\overline{y} = \overline{x}\sigma = \overline{z}\sigma = \overline{y}\sigma$.

LaG =

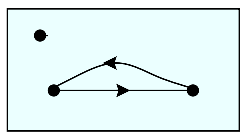


Remark 11.4.1 Notice that in all of these examples the functor La preserves the number of connected components. This is not a coincidence and the general case will be proved in theorem 13.1.3

`EXERCISE 11.4.1`

> Let $Rgraphs \xrightarrow{U} Graphs$ be the 'forgetful' functor from reflexive graphs into graphs. Compute LaG and RaG for

G =



11.5 The right adjoint to Γ : the functor B

In some cases the functor Γ has a right adjoint B, called *codiscrete* or chaotic. Thus, we have the diagram of adjoint functors

$$Set^{\mathbb{C}^{op}} \begin{array}{c} \xrightarrow{\Pi} \\ \xleftarrow{\Delta} \\ \xrightarrow{\Gamma} \\ \xleftarrow[B]{} \end{array} Set$$

$$\Pi \dashv \Delta \dashv \Gamma \dashv B$$

The following result gives necessary and sufficient conditions for the existence of this further adjoint

Theorem 11.5.1 (F.W. Lawvere) *The functor Γ has a right adjoint B iff at least one generic figure has a point, i.e., there is $\mathbb{1} \xrightarrow{p} h_C$ for some $C \in \mathbb{C}$.*

Proof.
Assume that B exists with $\Gamma \dashv B$. We then have a natural bijection

$$\frac{\Gamma X \longrightarrow S}{X \longrightarrow BS}$$

In particular for $X = h_C$, we have that B is forced on objects: in fact we have the natural equivalences

$$\begin{array}{c} C \dashrightarrow BS \\ \hline h_C \longrightarrow BS \\ \hline \Gamma(h_C) \longrightarrow S \end{array}$$

Thus $BS(C) = S^{\Gamma(h_C)}$. Similarly, B is forced on functions: the commutative triangles

$$\begin{array}{c}
\begin{tikzcd} & C \arrow[r, dashed, "\sigma"] & BS \\ C' \arrow[ur, "f"] \arrow[urr, dashed, "\sigma.f"'] & & \end{tikzcd} \\ \hline
\begin{tikzcd} & \Gamma(h_C) \arrow[r, "\sigma"] & S \\ \Gamma(h_{C'}) \arrow[ur, "\Gamma(h_f)"] \arrow[urr, "\sigma\circ\Gamma(h_f)"'] & & \end{tikzcd}
\end{array}$$

correspond to each other in the above natural bijection. Thus the action is forced to be $\sigma.f = \sigma \circ \Gamma(h_f)$

We have to check that $\Gamma \dashv B$.

Let $\Gamma(X) \xrightarrow{\Psi} S$ be given. Define

$$\Phi : X \longrightarrow BS$$

to be the natural transformation whose C-component is given by the function $\Phi_C : X(C) \longrightarrow BS(C) = S^{\Gamma(h_C)}$ that sends $\sigma \in X(C)$ into $\Phi_C(\sigma) : \Gamma(h_C) \longrightarrow S$ defined by $\Phi_C(\sigma)(p) = \Psi(\sigma \circ p)$ (with $p \in \Gamma(h_C)$).

Conversely, assume that $\Phi : X \longrightarrow BS$ is given. We define

$$\Psi : \Gamma X \longrightarrow S$$

on $p \in \Gamma X$ as follows: let $\mathbb{1} \xrightarrow{p_0} h_{C_0}$ be a point of a generic figure. Thus

$$\mathbb{1} \underset{p_0}{\overset{r_0}{\leftrightarrows}} h_{C_0}$$

Define $\Psi(p) = \Phi_{C_0}(pr_0)(p_0)$.

We leave to the reader the verification that these maps are natural and inverse to each other. Thus we have shown that $\Gamma \dashv B$ from the existence of a point of a generic figure. To show the other direction: assume that $\Gamma(h_C) = \emptyset$ for all C. Then $BS(C) = S^{\emptyset} = 1$ and thus $BS = \Delta 1$. Clearly this B cannot be a right adjoint of Γ.

Corollary 11.5.2 *Assume that B exists. Then the unit $Id \xrightarrow{\eta} \Gamma\Delta$ of the adjunction $\Delta \dashv \Gamma$ as well as the counit $\Gamma B \xrightarrow{\epsilon} Id$ of the adjunction $\Gamma \dashv B$ are isomorphisms.*

Proof.
Counit: Let S be a set. The equivalences

$$\frac{\frac{\frac{\mathbb{1} \xrightarrow{p} BS}{\Gamma\mathbb{1} \xrightarrow{p} S}}{1 \xrightarrow{p} S}}{p \in S}$$

show that ϵ_S is a bijection.

Unit: the unit is an isomorphism precisely when Δ is full and faithful, by proposition 11.1.2. We shall only prove fullness, leaving faithfulness to the reader.

Let $\Delta S \xrightarrow{\Phi} \Delta T$ be given. Take a generic figure C_0 with a point $\mathbb{1} \xrightarrow{p_0} h_{C_0}$ and define $f = \Phi_{C_0} : S \longrightarrow T$. To show that $\Phi = \Delta f$ we notice that there is a morphism from any generic figure A into C_0. Indeed, compose the unique map $h_A \xrightarrow{!_A} \mathbb{1}$ with the point $\mathbb{1} \xrightarrow{p_0} h_{C_0}$ and use Yoneda. The result follows from the fact that Φ is a natural transformation.$\Box$

As a consequence of the theorem, the functor B exists when the category of $\mathbb{C}$-sets satisfies the following condition:

A category $\mathbb{A}$ with both initial and terminal objects satisfies the *Nullstellensatz* iff every non initial object A has a point $\mathbb{1} \xrightarrow{p} A$. Thus the category $Sets^{\mathbb{C}^{op}}$ satisfies the *Nullstellensatz* iff every non-empty $\mathbb{C}$-set X has a point

$\mathbb{1} \xrightarrow{p} X$. Since the representable $\mathbb{C}$-sets (which are clearly inhabited) generate the category of $\mathbb{C}$-sets, $Sets^{\mathbb{C}^{op}}$ satisfies the Nullstellensatz iff every representable (or generic figure) has a point.

Examples

	Nullstellensatz
Sets	Yes
Bisets	No
Bouquets	No
Graphs	No
Rgraphs	Yes
Esets	No

The term 'Nullstellensatz' is the name given to the following basic theorem (due to D. Hilbert) on the existence of zeros of ideals (see e.g. [10], volume II p.429)

Theorem 11.5.3 (Nullstellensatz) *If K is an algebraically closed field, $A = K[X_1, X_2, \ldots, X_n]$ a ring of polynomials with coefficients in K and $I \subseteq A$ an ideal, then $1 \notin I$ iff there is a point $p = (a_1, a_2, \ldots, a_n) \in K^n$ such that $\forall f \in I\ f(p) = 0$.*

Since A is the free K-ring on n generators, the condition that $1 \notin I$ says that the ring A/I is not trivial. Furthermore, a point $p \in K^n$ which is a common zero of the polynomials in the ideal I may be identified with a ring homomorphism $A/I \longrightarrow K$. This terminology may be justified from the categorical point of view as follows: going over to the category of geometric *loci*, namely the dual of the category $\mathbb{K}$ of finitely generated K-algebras, such a map becomes a map in the opposite direction $K \longrightarrow A/I$,i.e., a point of A/I, since K is the terminal object in $\mathbb{K}^{op}$.

Corollary 11.5.4 *Assume that $Sets^{\mathbb{C}^{op}}$ satisfies the Nullstellensatz. Then B exists.*

Proof.
Obvious since every generic figure h_C is non-empty: it has the generic C-figure 1_C. Therefore every generic figure has a point.

The following table tells us for which of our examples this functor exists

	B exists
Sets	Yes
Bisets	No
Bouquets	Yes
Graphs	No
Rgraphs	Yes
Esets	No

Just as for the adjunction $\Delta \dashv \Gamma$, we can ask when is the unit (respectively the counit) of the adjunction $\Gamma \dashv B$ an isomorphism. The answer is given by the following

Proposition 11.5.5 *(1) The counit* $\Gamma B \xrightarrow{\epsilon} Id$ *is always an isomorphism*

(2) The unit $Id \xrightarrow{\eta} B\Gamma$ *is an isomorphism iff* $\mathbb{C} \simeq \mathbb{1}$

Proof.
(1) already done in corollary 11.5.2

(2) The equivalences

$$\frac{\dfrac{X \longrightarrow Y}{X \longrightarrow B\Gamma Y}}{\Gamma X \longrightarrow \Gamma Y}$$

show that Γ is full and faithful. By proposition 11.1.4 this is equivalent to $\mathbb{C} \simeq \mathbb{1}$. (Notice that the existence of B implies that $\mathbb{C}$ has at least one object.)

`EXERCISE 11.5.1`

(1) Find the mistake in the following argument. The equivalences

$$\frac{\dfrac{S \longrightarrow \Gamma BT}{\Delta S \longrightarrow BT}}{\Gamma \Delta S \longrightarrow T}$$

show that $\Gamma\Delta \dashv \Gamma B$. Since $\Gamma B \simeq Id$, then $\Gamma\Delta \simeq Id$. But this is not generally true. In fact this is true precisely when Δ is full and faithful or, equivalently, when $\Pi\mathbb{1} = 1$.

(2) Show that the counit $\Delta\Gamma F \xrightarrow{\epsilon_F} F$ has the following property: for every morphism from $\Delta\Gamma F$ into a codiscrete BX there is a unique $F \longrightarrow BX$ such that the diagram

$$\begin{tikzcd}
\Delta\Gamma F \arrow[r] \arrow[dr] & F \arrow[d] \\
& BX
\end{tikzcd}$$

is commutative

(3) Show the existence of a natural transformation

$$\Delta \longrightarrow B$$

Furthermore, show that it is injective iff every generic figure has a point.

Computation of $B : Sets \longrightarrow Rgraphs$

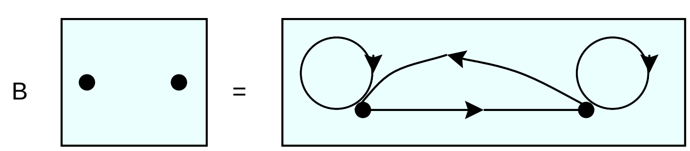
We shall compute BS (where S is a set) by extracting generic figures:

$$\begin{array}{c}
V \dashrightarrow BS \\ \hline
\Gamma h_V \longrightarrow S \\ \hline
1 \longrightarrow S \\ \hline
\text{elements of } S
\end{array}
\qquad
\begin{array}{c}
A \dashrightarrow BS \\ \hline
\Gamma h_A \longrightarrow S \\ \hline
1+1 \longrightarrow S \\ \hline
\text{couples of elements of } S
\end{array}$$

with the obvious incidence relations (left as exercise.)

Therefore, BS has as vertices the elements of the set S. Furthermore, for each couple (a, b) of elements of S there is a unique arrow whose source is a and whose target is b. Thus BS is the complete graph on the elements of S.

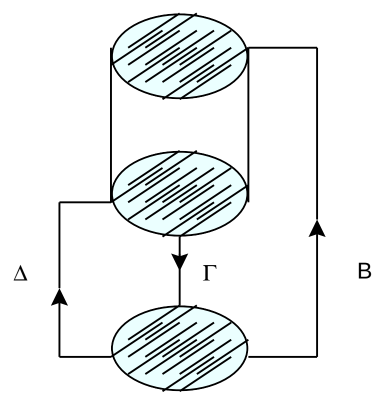
As an example:



WINDOW 11.5.1

Unity and identity of opposites

Lawvere has given a mathematical model for Hegel's notion of unity and identity of opposites which can best be described in the particular case of reflexive graphs. The category of sets is embedded in that of graphs in two different ways: via the functor Δ as discrete graphs (i.e., of the form ΔS) and via the functor B as codiscrete graphs (i.e., of the form BS). Although the notions of discrete and codiscrete are dual, the full subcategories of discrete graphs and codiscrete graphs are both equivalent to the category of sets, the identity of the title. Furthermore, both discrete and codiscrete are united in the category of all graphs. We can represent pictorially the situation as a cylinder whose top is isomorphic to the bottom. Although different they are united in the cylinder:



In the particular case of $\mathbb{C}$-sets we consider the bottom as the discrete $\mathbb{C}$-sets and the top as consisting of the codiscrete $\mathbb{C}$-sets with Γ as the common projection. In all cases $\Gamma\Delta \simeq Id \simeq \Gamma B$. Lawvere has further developed and applied these ideas to calculus and physics. Furthermore he has shown how Hegel Aufhebung can be formulated in this context, obtaining, as a consequence a theory of dimension in the context of $\mathbb{C}$-sets. The interested reader may consult [23] and [24].

12 Connectivity

In this section we define and study the notions of a connected $\mathbb{C}$-set and a connected component of a $\mathbb{C}$-set. This will allow us to prove the conjecture, formulated in section 11.3 for the particular case of graphs, that ΠX is indeed the set of connected components of a $\mathbb{C}$-set X.

12.1 Connected $\mathbb{C}$-sets

We start with a

Proposition 12.1.1 *Every $X \in Set^{\mathbb{C}^{op}}$ may be decomposed as $X = \bigsqcup_\alpha X_\alpha$, with $\Pi X_\alpha = 1$ for all α.*

Proof.
Since a small category is a graph and a graph may be expressed as the (disjoint) union of its connected components, $Fig(X)$ may be expressed as the coproduct of its connected components (which are in fact categories):

$$Fig(X) = \bigsqcup_{\alpha \in \Pi_0(Fig(X))} \alpha$$

where $\Pi_0(Fig(X))$ is the set of connected components of $Fig(X)$.

In more detail, define $\sim_X$ to be the equivalence relation on the set $\bigsqcup_{C\in\mathbb{C}} X(C)$ generated by the couples $(\sigma, \sigma.f)$, where $\sigma \in X(C)$ and $f : C' \longrightarrow C$. We let $\Pi_0(Fig(X)) = \bigsqcup_{C\in\mathbb{C}} X(C)/\sim_X$ be the set of equivalence classes and

$$[\ ] : Fig(X) \longrightarrow \Pi_0(Fig(X))$$

be the canonical map that sends σ into its equivalence class $[\sigma]$. The elements of $\Pi_0(Fig(X))$ are the connected components of the category $Fig(X)$. Let X_α to be the sub $\mathbb{C}$-set of X defined as follows:

$$X_\alpha(C) = \{\sigma \in X(C) | [\sigma] = \alpha\}$$

Since $X_\alpha = colim\ U_{X_\alpha}$ and Π preserves colimits, $\Pi X_\alpha = colim\ \Pi \circ U_{X_\alpha}$. But, as we saw, $\Pi \circ U_{X_\alpha} = \delta 1$, where $\delta 1 : Fig(X_\alpha) \longrightarrow Sets$ is the functor whose

value is constantly 1. Clearly $Fig(X_\alpha)$ is connected and this implies that $\Pi X_\alpha = 1$. Furthermore,

$$X = \coprod_{\alpha \in \Pi_0(Fig(X))} X_\alpha.$$

The $\mathbb{C}$-sets X for which $\Pi X = 1$ have a clear geometrical interpretation:

Proposition 12.1.2 *The following conditions are equivalent for a $\mathbb{C}$-set X:*

(1) $X \neq 0$ *and whenever* $X = X_1 + X_2$*, then* $X = X_1$ *or* $X = X_2$

(2) $X \neq 0$ *and every morphism* $X \xrightarrow{f} F + G$ *factors either through* $F \xrightarrow{i_F} F + G$ *or* $G \xrightarrow{i_G} F + G$*. Diagrammatically:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed] & F + G \\ & F \arrow[u, "i_F"'] \end{tikzcd} \quad \textit{or} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed] & F + G \\ & G \arrow[u, "i_G"'] \end{tikzcd}$$

(3) $\Pi X = 1$

Proof.
Let us first notice that for a morphism $X \xrightarrow{f} F + G$, f factors through i_F iff $f^*F = X$.

It is easy to see that (1) and (2) are equivalent: in fact (1) is the particular case of (2) when $f = 1_X$. Conversely, assume (1) and let $f : X \longrightarrow F + G$. Therefore, $X = f^*F + f^*G$. Since $X \neq 0$, at least one of these summands is not zero, say f^*F. By (1), $X = f^*F$, i.e., X factors through i_F.

(1)$\rightarrow$(3) : By proposition 12.1.1, we can write $X = \coprod_\alpha X_\alpha$ with $\Pi X_\alpha = 1$, for each α. Since $X \neq 0$, we may conclude, as before, that one of the X's is not zero, say X_{α_0}. But $X = X_{\alpha_0} + (\coprod_{\alpha \neq \alpha_0} X_i)$. By (1), $X = X_{\alpha_0}$ and this implies that $\Pi X = \Pi X_{\alpha_0} = 1$.

(3)$\rightarrow$(1) : Assume that $X = X_1 + X_2$ with $X_1 \neq 0$. Since Π has a right adjoint, it preserves coproducts and hence $\Pi X = \Pi X_1 + \Pi X_2$. By (3), $\Pi X = 1$. But, as easily seen, $\Pi X = 0$ iff $X = 0$. Thus $\Pi X_1 = 1$ and consequently $\Pi X_2 = 0$, i.e., $X_2 = 0$. In other words, $X = X_1$.

Remark 12.1.3 The proof shows that

(1)' $X \neq 0$ and whenever $X = \bigsqcup_\alpha X_\alpha$, then $X = X_\alpha$ for some α

is equivalent to any of the above.

We define X to be *connected* iff X satisfies any of the above (equivalent) statements.

We say that C is a *connected component* of X iff C is connected and there is a $\mathbb{C}$-set Y such that $X = C + Y$. Thus a connected component of X is a connected complemented sub $\mathbb{C}$-set of X.

We are now in a position to formulate the following

Proposition 12.1.4 *Every $X \in Sets^{\mathbb{C}^{op}}$ can be expressed as the coproduct of its connected components. Furthermore X has $\Pi_0(Fig(X))$ connected components.*

Proof.
By proposition 12.1.1 X may be written as $X = \bigsqcup_\alpha X_\alpha$ with $\Pi X_\alpha = 1$ for each α. Thus, X is a coproduct of connected components, by the very definition of a connected component. Let us show that every connected component C appears in the above coproduct or, more precisely, that $C = X_\alpha$ for some α. By distributivity $C = \bigsqcup_\alpha (X_\alpha \cap C)$. Since C is connected, $C = X_\alpha \cap C$ for some α, i.e., $C \hookrightarrow X_\alpha$. Furthermore, C has a complement in X_α (the intersection of the complement of C in X with X_α). Since X_α is connected, $C = X_\alpha$.

This is the missing link to prove the conjecture of section 11.3

Proposition 12.1.5 *For every $X \in Sets^{\mathbb{C}^{op}}$,*

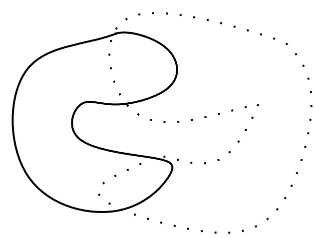
$$\Pi X = \bigsqcup_{\alpha \in \Pi_0(Fig(X))} 1$$

Thus, there is a bijection between ΠX and the set of connected components of X.

Proof.
Indeed, from $X = \bigsqcup\{C : C \text{ is a connected component of } X\}$, we obtain at once that $\Pi X = \bigsqcup_C 1$. Thus ΠX has as many elements as there are connected components of X.

We turn now to the properties of connected $\mathbb{C}$-sets.

We do not expect the intersection of two connected sub $\mathbb{C}$-sets of a connected $\mathbb{C}$-set to be connected, since this is not true for the topological space $\mathbb{R}^2$, as the following example shows:



On the other hand we have

Proposition 12.1.6 *The quotient of a connected $\mathbb{C}$-set is connected.*

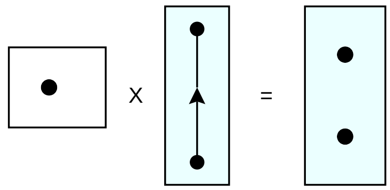
Proof.
Assume that

$$X \overset{q}{\twoheadrightarrow} Y$$

is a quotient (i.e. $\exists_q X = Y$), X is connected and $Y \xrightarrow{f} F+G$ is a morphism. We show that Y is connected, namely that $Y \neq 0$ and f factors through i_F or i_G. But since X is connected, $f \circ q$ factors either through i_F or i_G. Suppose the first alternative. Then $(f \circ q)^* F = X$, i.e., $q^*(f^*F) = X$. Therefore, $\exists_q q^*(f^*F) = \exists_q X$. Since q is a quotient, $\exists_q q^* Y = Y$ and hence $f^*F = Y$, i.e., f factors through i_F.

Notice that $Y \neq 0$ since $X \neq 0$ (given that X is connected and $X \xrightarrow{q} Y$ is a quotient). $\square$

Contrary to what happens in topological spaces, the product of two connected $\mathbb{C}$-sets is not connected in general. An example in *Graphs* is provided by the product $h_V \times h_A = \mathbb{1} + \mathbb{1}$:



However, to show that products of connected are connected it suffices to test representables:

Theorem 12.1.7 *The following are equivalent:*

(1) The product ot two connected is connected

(2) The product of two generic (figures) is connected

Proof.
(1)→*(2)*: immediate, since generic figures are connected (see section 11.3)

(2)→*(1)*: we first show that the product of a connected and a generic is connected. The proof depends on the following

Lemma 12.1.8 *Assume that the product of two representables is connected. Then the canonical morphism*

$$X^{h_C} + Y^{h_C} \longrightarrow (X+Y)^{h_C}$$

is an isomorphism.

Proof.
We have the following equivalences:

$$\frac{C' \dashrightarrow (X+Y)^{h_C}}{h_{C'} \times h_C \longrightarrow (X+Y)}$$

Since $h_{C'} \times h_C$ is connected, this arrow factors through either $X \xrightarrow{i_X} (X+Y)$ or $Y \xrightarrow{i_Y} (X+Y)$. Assume the first alternative. By exponential adjointness, we obtain a map $h_{C'} \longrightarrow X^{h_C}$. By composing with the canonical inclusion $i_{X^{h_C}} : X^{h_C} \longrightarrow (X^{h_C} + Y^{h_C})$ we obtain a map $h_{C'} \longrightarrow (X^{h_C} + Y^{h_C})$. The conclusion follows from the equivalence

$$\frac{h_{C'} \longrightarrow (X^{h_C} + Y^{h_C})}{C' \dashrightarrow (X^{h_C} + Y^{h_C})}$$

General case: we proceed as before, by first showing the following

Lemma 12.1.9 *Assume that the product of a connected by a representable is connected. Then if X is connected, the canonical morphism*

$$F^X + G^X \longrightarrow (F+G)^X$$

is an isomorphism.

Proof.
Completely analogous to the preceding one.□

Assume that X and Y are connected and let $X \times Y \xrightarrow{f} (F+G)$ be an arbitrary morphism. Then we have the following equivalences:

$$\frac{\dfrac{X \times Y \longrightarrow F+G}{X \longrightarrow (F+G)^Y}}{X \longrightarrow F^Y + G^Y}$$

Since X is connected this map factors either through $F^Y \hookrightarrow F^Y + G^Y$ or through $G^Y \hookrightarrow F^Y + G^Y$. If we assume the first alternative, we obtain a factorization of the original map f through the inclusion $F \hookrightarrow F+G$. Thus $X \times Y$ is connected. □

Proposition 12.1.10 *The product of two connected is connected iff* Π *preserves binary products*

Proof.
$\rightarrow$: Let X and Y be $\mathbb{C}$-sets. Decompose X and Y as coproducts of their connected components: $X = \bigsqcup_\alpha X_\alpha$ and $Y = \bigsqcup_\beta Y_\beta$. Then $X \times Y = \bigsqcup_{(\alpha,\beta)} X_\alpha \times Y_\beta$ (by distributivity). Each summand is a product of connected $\mathbb{C}$-sets and hence it is connected. Applying Π to this product,

$$\Pi(X \times Y) = \bigsqcup_{(\alpha,\beta)} 1 = \Pi(X) \times \Pi(Y).$$

$\leftarrow$: Let X and Y be connected. Then $\Pi(X \times Y) = \Pi X \times \Pi Y = 1$.

There is a case, simple to test, when products of connected is connected, namely when the category of $\mathbb{C}$-sets satisfies the Nullstellensatz

Theorem 12.1.11 *Assume that* $Sets^{\mathbb{C}^{op}}$ *satisfies the Nullstellensatz. Then*

(1) The terminal object $\mathbb{1}$ is connected

(2) The product of two connected $\mathbb{C}$-sets is connected

Proof.
(1): Assume that $\mathbb{1} = F + G$ with F non-empty. By the Nullstellensatz, F has a point $\mathbb{1} \xrightarrow{p} F$. Since F is a sub $\mathbb{C}$-set of $\mathbb{1}$, it has at most one F-figure, for each F. On the other hand, the existence of p implies that F has at least one F-figure for each F. In other words, $F = \mathbb{1}$ and so $\mathbb{1}$ is connected.

(2):(suggested by S. Schanuel). We argue by contradiction. Assume that $X \times Y$ may be decomposed as $X \times Y = F + G$ with F and G non-empty (i.e., having al least one generic figure each). By the Nullstellensatz, both F and G have points

$$\mathbb{1} \xrightarrow{(p,p')} F \hookrightarrow h_C \times h_{C'}$$

$$\mathbb{1} \xrightarrow{(q,q')} G \hookrightarrow h_C \times h_{C'}$$

The decomposition $X \times Y = F + G$ may be represented by the morphism $f : X \times Y \longrightarrow \mathbb{1} + \mathbb{1}$ defined in the obvious way:

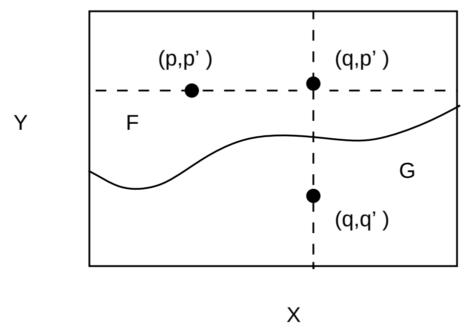
$$f_C(\sigma, \xi) = \begin{cases} 0 & \text{if } (\sigma, \xi) \in F(C) \\ 1 & \text{if } (\sigma, \xi) \in G(C) \end{cases}$$

We recall that a point $\mathbb{1} \xrightarrow{p} h_C$ is a natural transformation, i.e. we have an equivalence

$$\begin{array}{c} \mathbb{1} \xrightarrow{p} h_C \\ \hline \end{array}$$

$$\begin{tikzcd}
C' & & h_C(C') \arrow[dd, "(\,).f"] & \\
 & 1 \arrow[ur, "p_{C'}"] \arrow[dr, "p_{C''}"'] & & p_{C'} \circ f = p_{C''} \\
C'' \arrow[uu, "f"] & & h_C(C'') &
\end{tikzcd}$$

The following picture may help the reader to understand the proof:



Let $H_{(p,p')} \hookrightarrow X \times Y$ be the 'horizontal line' going trough (p, p'). Thus,

$$H_{(p,p')}(C) = \{(\sigma, \xi) \in X(C) \times Y(C) : \xi = p'_C\}.$$

Clearly this line is isomorphic to X. Since X is connected, the restriction of f to $H_{(p,p')}$ is constant.

Similarly, the restriction of f to $V_{(q,q')}$ is again a constant, where $V_{(q,q')}$ is the 'vertical line' going through (q, q'). But these lines intersect at (q, p'), showing that these constants coincide, a contradiction. Further details are left to the reader.

Corollary 12.1.12 *Assume that* $Sets^{\mathbb{C}^{op}}$ *satisfies the Nullstellensatz. Then*

(i) $\Pi \mathbb{1} = 1$

(ii) $\Pi(X \times Y) = \Pi X \times \Pi Y$.

12.2 Connectivity of Ω

One of the most surprising results of the categories of $\mathbb{C}$-sets is the fact that, contrary to the case *Sets*, Ω may be connected. First a

Lemma 12.2.1 *Let H be a Heyting algebra. Then the following are equivalent:*

(1) $\forall a \in H\ (\neg a \vee \neg\neg a) = 1$

(2) $\forall a, b \in H\ \neg(a \wedge b) = \neg a \vee \neg b$ *(De Morgan law)*

Proof.
Left as exercise.

A Heyting algebra satisfying any of these (equivalent) conditions is called a *Stone algebra* .

Theorem 12.2.2 (P.T. Johnstone) *The following conditions are equivalent for a category of $\mathbb{C}$-sets*

(1) For every $\mathbb{C}$-set X $Sub(X)$ is a Stone algebra

(2) The Ore condition holds for $\mathbb{C}$: every diagram

$$\begin{tikzcd} & C & \\ C_1 \arrow[ur] & & C_2 \arrow[ul] \end{tikzcd}$$

may be completed to a commutative diagram

$$\begin{tikzcd} & C & \\ C_1 \arrow[ur] & & C_2 \arrow[ul] \\ & C_3 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Proof.
(1) → (2): Let

$$\begin{tikzcd} & C & \\ C_1 \arrow[ur, "f"] & & C_2 \arrow[ul, "g"'] \end{tikzcd}$$

be a diagram in $\mathbb{C}$.

We define $< f >$ to be the following subfunctor of h_C :

$$< f > (C') = \{C' \xrightarrow{h} C \in \mathbb{C} \mid h \text{ factors through} f\}$$

Clearly the diagram can be completed if and only if $< f > \wedge < g > \neq 0$.

Let us suppose that

$$\begin{tikzcd} & C & \\ C_1 \arrow[ur, "f"] & & C_2 \arrow[ul, "g"'] \end{tikzcd}$$

cannot be completed. Then $< f > \wedge < g >= 0$, and

$$\neg(< f > \wedge < g >) = \neg < f > \vee \neg < g >= h_C$$

because of *(1)*. So $1_C \in \neg < f > (C)$ or $1_C \in \neg < g > (C)$. Assume the first alternative. Then $C \Vdash \neg < f > [1_C]$, namely, for all $C' \xrightarrow{h} C$ $C' \nVdash < f > [h]$. Saying that $C' \Vdash < f > [h]$ is the same as saying that h factors through f. But for $h = f$ this is false. We obtain a contradiction so, $< f > \wedge < g > \neq 0$.
(2) $\rightarrow$ *(1)*: Let $A \hookrightarrow X$. We have to show that

$$X = \neg_X A \vee \neg_X \neg_X A$$

Let $\sigma \in X(C)$, $\sigma \notin (\neg_X A)(C)$. We have the following equivalences

$$\begin{array}{c} C \nVdash \neg A[\sigma] \\ \hline \textit{not } (\forall C' \xrightarrow{f} C \ C' \nVdash A[\sigma, f]) \\ \hline \exists C' \xrightarrow{f} C \ C' \Vdash A[\sigma, f] \end{array}$$

We claim that $C \Vdash \neg\neg A[\sigma]$, namely that

$$\forall C'' \xrightarrow{g} C \ \ \exists C''' \xrightarrow{h} C'' \ \ C''' \Vdash A[\sigma, (g \circ h)].$$

Let $C'' \xrightarrow{g} C$ be given. By the Ore condition the diagram

$$\begin{tikzcd} & C & \\ C' \arrow[ur, "f"] & & C'' \arrow[ul, "g"'] \end{tikzcd}$$

can be completed to the diagram

$$\begin{tikzcd} & C & \\ C' \arrow[ur, "f"] & & C'' \arrow[ul, "g"'] \\ & C''' \arrow[ul, "k"] \arrow[ur, "h"'] & \end{tikzcd}$$

But $C' \Vdash A[\sigma.f]$. Hence $C''' \Vdash A[(\sigma.f).k]$. But

$$(\sigma.f).k = \sigma.(f \circ k) = \sigma.(g \circ h).$$

This implies that $C''' \Vdash A[\sigma.(g \circ h)]$. $\square$

The reader can verify the Ore condition in the examples. The following table summarizes the results

	Ore condition for $\mathbb{C}$
Sets	Yes
Bisets	Yes
Bouquets	Yes
Graphs	No
Rgraphs	No
Esets	Yes

The following result gives necessary and sufficient conditions for Ω to be connected

Theorem 12.2.3 (F.W. Lawvere) *The following are equivalent for a category of $\mathbb{C}$-sets*

(1) Ω is connected

(2) $\mathbb{1}$ is connected and the Ore condition for $\mathbb{C}$ fails

(3) $\mathbb{1}$ is connected and Sub(X) is not a Stone algebra, for some $\mathbb{C}$-set X

(4) There is a connected $\mathbb{C}$-set X such that $\mathbb{1} + \mathbb{1} \hookrightarrow X$ ('Any two points may be connected')

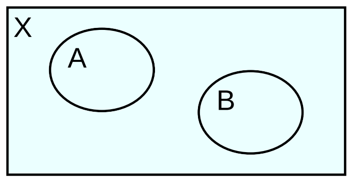
Proof.
(2) and *(3)* are equivalent by Johnstone's theorem

(1) $\to$ *(4)*: obvious by noticing that $\top : \mathbb{1} \hookrightarrow \Omega$ and $\bot : \mathbb{1} \hookrightarrow \Omega$ combine to give $\mathbb{1} + \mathbb{1} \hookrightarrow \Omega$.

(4) $\to$ *(3)*: Let X be a connected $\mathbb{C}$-set such that $\mathbb{1} + \mathbb{1} \hookrightarrow X$. Consider the diagram

$$\begin{tikzcd}
\mathbb{1} \arrow[r, hook, "i_1"] & \mathbb{1}+\mathbb{1} \arrow[d, hook] \\
& X \arrow[ul]
\end{tikzcd}$$

By applying the functor Π we conclude that $\Pi\mathbb{1} = 1$, showing that $\mathbb{1}$ is connected. Let us suppose that the law of De Morgan holds for this X. In particular, if A and B are the two disjoint copies of $\mathbb{1}$ inside X, $A \wedge B = 0$:



Therefore $\neg_X A \vee \neg_X B = X$ (by the De Morgan law). But $B \hookrightarrow \neg_X A$ and so $\neg_X A \neq 0$. By the same argument, $\neg_X B \neq 0$. Since X is connected, this is a contradiction.

(2) $\rightarrow$ *(1)*: We first prove a

Lemma 12.2.4 *Assume that $\mathbb{1}$ is connected. Then Ω has at most two connected components.*

Proof.
Notice first that

(i) All generic figures of Ω of the form $\top_C \hookrightarrow h_C$ are in one connected component, the 'connected component of the truth'

(ii) All generic figures of Ω of the form $\perp_C \hookrightarrow h_C$ are in one connected component, the 'connected componenent of the false'

(iii) All generic figures of Ω of the form $\sigma \hookrightarrow h_C$, with $\sigma \neq 0$, are in the connected component of the truth

The proof of the first two is fairly easy, since $\top$ and $\perp$ are preserved by pullbacks. Take (i). Recall that for a $\mathbb{C}$-set X, $\sim_X$ on $\coprod_C X(C)$, the relation that holds for two generic figures of X when they are in the same component, was defined to be the smallest equivalence relation generated by the couples $(\sigma, \sigma.f)$ (with σ a C-figure of X and $f : C' \longrightarrow C$). From this description it follows that $((\top_C \hookrightarrow h_C), (\top_{C'} \hookrightarrow h_{C'}))$ is a generating couple for $\sim_\Omega$ iff

$$((C \dashrightarrow \mathbb{1}),\ (C' \dashrightarrow \mathbb{1}))$$

is a generating couple for $\sim_{\mathbb{1}}$. This, in turn, implies the equivalence

$$\frac{(\top_C \hookrightarrow h_C) \sim_\Omega (\top_{C'} \hookrightarrow h_{C'})}{(C \dashrightarrow \mathbb{1}) \sim_{\mathbb{1}} (C' \dashrightarrow \mathbb{1})}$$

Since $Fig(\mathbb{1})$ has one connected component, (i) follows.

To show (iii), let $(\sigma' \hookrightarrow h_{C'})$ with $\sigma' \neq 0$. Then

$$\exists\, C'' \overset{g}{\dashrightarrow} \sigma' \hookrightarrow h_{C'}$$

In this case

$$\begin{tikzcd} \sigma' \arrow[r, hook] & h_{C'} \\ \top_{C''} = h_{C''} \arrow[u] \arrow[r, equal] & h_{C'} \arrow[u, "(\,)\cdot g"'] \end{tikzcd}$$

Thus

$$(\sigma' \hookrightarrow h_C) \sim (\top_{C''} = h_{C''})$$

This concludes the proof of the lemma. $\square$

Assume that the diagram

$$\begin{tikzcd} & C & \\ C_1 \arrow[ur, "f"] & & C_2 \arrow[ul, "g"'] \end{tikzcd}$$

cannot be completed. By looking at the pullbacks

$$\begin{tikzcd} < f > \arrow[r, hook] & h_C \\ \top_{C_1} = h_{C_1} \arrow[u] \arrow[r, equal] & h_{C_1} \arrow[u, "h_f"'] \end{tikzcd} \qquad \begin{tikzcd} < f > \arrow[r, hook] & h_C \\ \bot_{C_2} = 0 \arrow[u] \arrow[r, hook] & h_{C_2} \arrow[u, "h_g"'] \end{tikzcd}$$

we deduce at once that $(\top_{C_1} \hookrightarrow h_{C_1}) \sim (\bot_{C_2} \hookrightarrow h_{C_2})$.

So there is only one connected component in $Fig(\Omega)$, i.e., Ω is connected.
$\square$

WINDOW 12.2.1

General versus particular toposes

It is an unexpected result that the category of reflexive graphs is qualitatively quite different from that of graphs, as Lover realized. In some non set-theoretical sense the first is a 'gros' or 'general' topos, whereas the second is a 'petit' or 'particular' topos. Roughly, a 'gros' topos is one which consists of all 'spaces' of a given kind, whereas a 'petit' topos is one consisting of all variable discrete sets varying continuously over a particular space (such as sheaves over a topological space). As an example of the qualitative difference between the categories of reflexive graphs and graphs: the functor Π preserves products for the first, but not for the second. The category of reflexive graphs has played an important role in the discovery of axioms for a topos to be a topos of spaces. This would allow us to make arguments of the type: let X and Y be spaces, without specifying whether they are topological spaces, smooth manifolds, measure spaces or other kind of spaces. This program has not yet been completed, although important partial results have been obtained.

For more on this subject, the reader may consult [25] .

`EXERCISE 12.2.1`

(1) Show that Ω has at least as many connected components as $\mathbb{1}$ and at most twice that number. Give examples in which both bounds are attained.

(2) Let $\mathbb{I}$ be a small connected category and let $\delta 1 : \mathbb{I} \longrightarrow Set$ be the functor whose value is constantly 1. Show that

$$colim\ \delta 1 = 1$$

(3) Show the following statements for $Sets^{\mathbb{C}^{op}}$: Δ is full and faithful iff $\mathbb{1}$ is connected iff $\mathbb{C}$ is connected

(4) Prove that X is connected iff $Sets^{\mathbb{C}^{op}}(X, \mathbb{1}+\mathbb{1}) = 2$.

(5) Assume that some representable in $Sets^{\mathbb{C}^{op}}$ has a point. If $\Gamma\Omega = 2$, show that the Nullstellensatz holds.

13 Geometric morphisms (bis)

In the cases studied up to now, the forgetful functor U can be obtained from another functor between the categories of generic figures. For instance

$$Graphs \xleftarrow{U} Bouquets$$

can be obtained from the functor

$$(V \underset{t}{\overset{s}{\rightrightarrows}} A) \xrightarrow{u} (V \xrightarrow{v} L)$$

defined by $uV = V$, $uA = L$ and $u(s) = u(t) = v$.

In fact, $U = u^*$, i.e., the functor whose value on a bouquet X is the graph $X \circ u$.

In the case of $Esets$ and $Graphs$

$$Graphs \xleftarrow{U} Esets$$

is obtained as $U = u^*$, where u is the functor

$$(V \underset{t}{\overset{s}{\rightrightarrows}} A) \xrightarrow{u} (* \xrightarrow{\sigma} *)$$

defined by $uV = uA = *$, $u(s) = 1$, $u(t) = \sigma$.

13.1 Geometric morphisms: general case

More generally given two categories $\mathbb{C}$ and $\mathbb{D}$ and a functor

$$\mathbb{C} \xrightarrow{u} \mathbb{D}$$

we obtain

$$Sets^{\mathbb{C}^{op}} \xleftarrow{u^*} Sets^{\mathbb{D}^{op}}$$

Proposition 13.1.1 *The functor u^* has both a left and a right adjoint:*

$$u_! \dashv u^* \dashv u_*$$

Proof.
We start with the following

Lemma 13.1.2 *The functor $Sets^{\mathbb{C}^{op}} \xleftarrow{u^*} Sets^{\mathbb{D}^{op}}$ preserves colimits*

Proof.
We have the following equivalences for a functor $\mathbb{I} \xrightarrow{F} Sets^{\mathbb{D}^{op}}$

$$\begin{array}{c}
u^*(colim\ F) \longrightarrow X \\ \hline
(colim\ F) \circ u \longrightarrow X \\ \hline
colim\ (F \circ u) \longrightarrow X \\ \hline
F \circ u \longrightarrow \Delta X \\ \hline
u^* F \longrightarrow \Delta X \\ \hline
colim\ u^* F \longrightarrow X
\end{array}
\qquad
\begin{array}{c}
\text{definition of } u^* \\
\text{colimits are computed point-wise} \\
colim \dashv \Delta \\
\text{definition of } u^* \\
colim \dashv \Delta
\end{array}$$

Hence $u^*(colim\ F) = colim\ u^* F$

We notice that the definition of $u_* X$ is forced on both objects and maps:

$$\begin{array}{c}
D \overset{\sigma}{\dashrightarrow} u_* X \\ \hline
u^* h_D \xrightarrow{\sigma} X
\end{array}$$

$$\begin{array}{c}
\begin{tikzcd}
 & D \arrow[r, dashed, "\sigma"] & u_* X \\
D' \arrow[ur, "g"] \arrow[urr, dashed, "\sigma . g"'] & &
\end{tikzcd} \\ \hline
\begin{tikzcd}
 & u^* h_D \arrow[r, "\sigma"] & X \\
u^* h_{D'} \arrow[ur, "u^* h_g"] \arrow[urr, "\sigma \circ u^* h_g"'] & &
\end{tikzcd}
\end{array}$$

To check that $u^* \dashv u_*$, let $X \in Sets^{\mathbb{C}^{op}}$ and $Y \in Sets^{\mathbb{D}^{op}}$. Then we have the following equivalences (recalling that $Y = colim\ U_Y$):

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Y \longrightarrow u_* X}{colim\ U_Y \longrightarrow u_* X}}{U_Y \longrightarrow \Delta u_* X}}{u^* \circ U_Y \longrightarrow \Delta X}}{colim\ (u^* \circ U_Y) \longrightarrow X}}{u^* colim\ U_Y \longrightarrow X}}{u^* Y \longrightarrow X}$$

The only non-trivial equivalence occurs in the third line and is left to the reader.

To define $u_!$ we proceed as we did for Π : replace the generic figures C of the blueprint of X by $u(C)$ and make the glueing indicated by that blueprint. More precisely,

$$u_! X = colim\ u \circ U_X$$

Notation: We have used the notation 'U_X' to denote either of two functors: the functor $Fig(X) \longrightarrow Sets^{\mathbb{C}^{op}}$ that sends

$$C \overset{\sigma}{\dashrightarrow} X \longmapsto h_C$$

and the functor $Fig(X) \longrightarrow \mathbb{C}$ that sends

$$C \overset{\sigma}{\dashrightarrow} X \longmapsto C$$

Since the first factors through the second (by identifying C with h_C) we hope that this does not create confusion.

To show that $u_! \dashv u^*$, we shall write '$colim_i F_i$' for '$colim\ F$', where $F : \mathbb{I} \longrightarrow \mathbb{A}$. Although somewhat imprecise, this notation is highly suggestive. In particular, instead of '$X = colim\ U_X$', we shall write '$X = colim_{C-->X} h_C$' and the definition of $u_!$ is given by $u_! X = colim_{C-->X} u(C)$

Using this notation, we check easily the following equivalences:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{u_! X \longrightarrow Y}{(colim_{C-->X} h_{u(C)}) \longrightarrow Y}}{\{h_{u(C)} \longrightarrow Y\}_C\ cocone}}{\{h_C \longrightarrow u^* Y\}_C\ cocone}\quad \text{(Yoneda)}}{(colim_{C-->X} h_C) \longrightarrow u^* Y}}{X \longrightarrow u^* Y}$$

From here we can recover the same equivalences, written this time in the official notation:

$$\frac{\frac{\frac{\frac{\frac{u_! X \longrightarrow Y}{colim\ u \circ U_X \longrightarrow Y}}{u \circ U_X \longrightarrow \Delta Y}}{U_X \longrightarrow \Delta u^* Y}}{colim\ U_X \longrightarrow u^* Y}}{X \longrightarrow u^* Y}$$

Thus, we obtain

$$u_! X(D) = colim_{\{h_C \to X\}} \mathbb{D}(D, u(C))$$

Example: geometric morphisms between presheaves and families

If $\mathbb{C}$ is any category, we let $|\mathbb{C}|$ to be the category that has the same objects of $\mathbb{C}$, but only identities and let $|\mathbb{C}| \xrightarrow{u} \mathbb{C}$ be the functor that sends objects into themselves (and identities into identities, as befits any functor). The functor u defines a diagram

$$\begin{tikzcd} Sets^{|\mathbb{C}|} \arrow[r, shift left=2, "u_!"] \arrow[r, shift right=2, "u_*"'] & Sets^{\mathbb{C}^{op}} \arrow[l, "u^*"'] \end{tikzcd}$$

The category $Sets^{|\mathbb{C}|}$ is simply the category of families indexed by the set of objects of $\mathbb{C}$, while u^* takes a $\mathbb{C}$-set X into the underlying family $(X(C))_C$. To compute its adjoints we take as generic figures of $Sets^{|\mathbb{C}|}$ the representables, i.e., the families h^0_C where

$$h^0_C(C') = \begin{cases} \emptyset & \text{if } C' \neq C \\ 1 & \text{if } C' = C \end{cases}$$

Notice that we have the following natural bijections (by Yoneda) which we view as identifications

$$\frac{h^0_C \xrightarrow{\sigma} A}{\sigma \in A_C}$$

To compute $u_!$, we notice that

$$A = \bigsqcup \{h^0_C | \sigma \in A_C\}$$

(corresponding to the fact that presheaves are obtained by glueing representables).

Thus, $u_!(A) = \bigsqcup\{u_! h^0_C | \sigma \in A_C\}$ and we need only compute $u_!$ of a representable:

$$\begin{array}{c} u_! h^0_C \xrightarrow{x} X \\ \hline h^0_C \xrightarrow{x} u^* X \\ \hline x \in X(C) \\ \hline h_C \xrightarrow{x} X \end{array}$$

Hence, $u_! h^0_C = h_C$ and we can write

$$u_! A = \bigsqcup\{h_C | \sigma \in A_C\}$$

To understand this formula better, let us compute the C_1-figures of $u_! A$:

$$\tau \in u_! A(C_1) \ \text{ iff } \ \exists! C \ \tau = (f, \sigma), \ C_1 \xrightarrow{f} C, \ \sigma \in A_C$$

The action by a change of figure $C_2 \xrightarrow{g} C_1$ is given by $(f, \sigma).g = (f \circ g, \sigma)$.

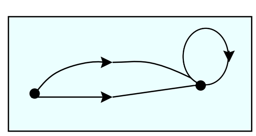
To compute u_* is simpler. In fact we have the following equivalences

$$\begin{array}{c} C \dashrightarrow u_* A \\ \hline u^* h_C \longrightarrow A \\ \hline h_C \circ u \longrightarrow A \\ \hline (h_C(C') \longrightarrow A_{C'})_{C'} \end{array}$$

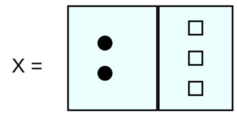
To illustrate this example, take $2 \xrightarrow{u} \left(V \underset{t}{\overset{s}{\rightrightarrows}} A \right)$ defined by $uP = V$ and $uS = A$. This functor gives rise to the diagram

$$\begin{tikzcd} Bisets \arrow[r, shift left=2, "u_!"] \arrow[r, shift right=2, "u_*"'] & Graphs \arrow[l, "u^*"'] \end{tikzcd}$$

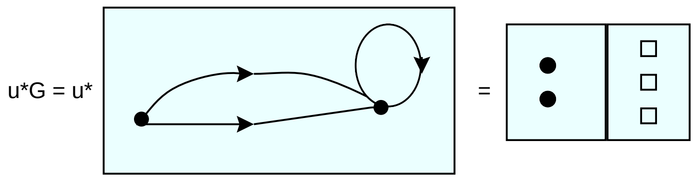
For

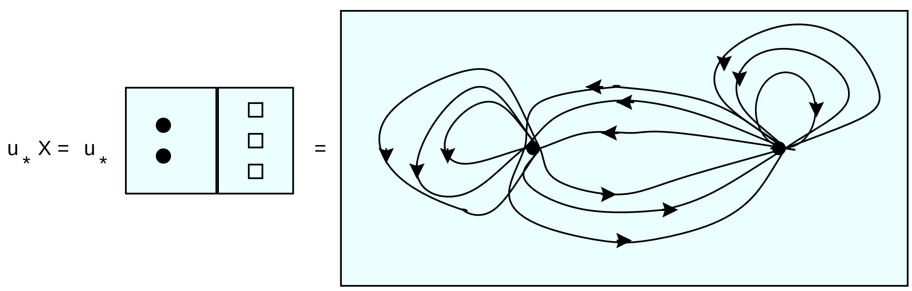
G = 

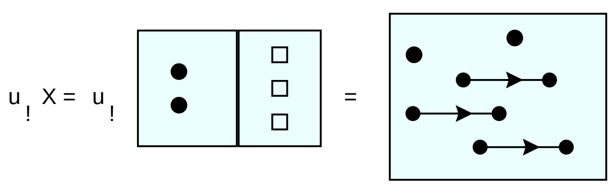
and



we have



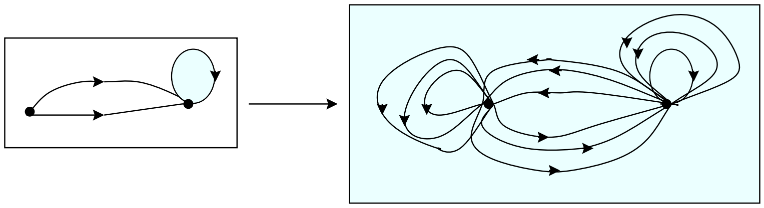




Units and counits are given by the following morphisms that the reader may easily describe:

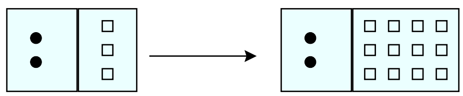
Unit of $u^* \dashv u_*$:

$G \xrightarrow{\eta} u_* u^* G =$



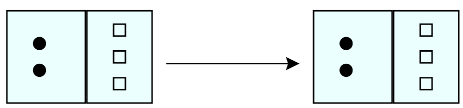
Counit of $u^* \dashv u_*$:

$u^* u_* X \xrightarrow{\epsilon} X =$



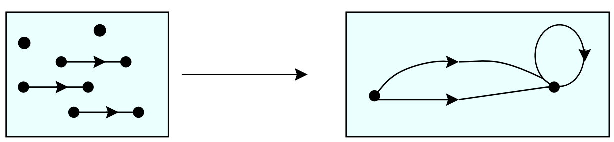
Unit of $u_! \dashv u^*$:

$X \xrightarrow{\eta} u^* u_! X =$



Counit of $u_! \dashv u^*$:

$u_! u^* G \xrightarrow{\epsilon} G =$



Notice that $u_! u^* G$ is a 'deconstruction' or 'deployment' of G whereas ϵ indicates the required identifications to obtain G.

`EXERCISE 13.1.1`

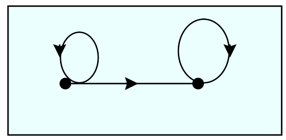
The functor

$$\Big(V \overset{s}{\underset{t}{\rightrightarrows}} A \Big) \xrightarrow{u} \Big(* \circlearrowleft \sigma \Big)$$

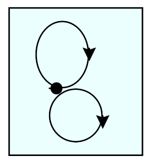
defines a diagram

$$\begin{tikzcd}
Esets \arrow[r, shift left=3, "u_!"] \arrow[r, shift right=3, "u_*"'] & Graphs \arrow[l, "u^*"']
\end{tikzcd}$$

Compute u_*G and $u_!G$ in the following two cases, when

G = 

and

G = 

In the examples, the functor $u_!$ preserves the number of connected components. This is not a coincidence. In fact

Proposition 13.1.3 *The diagram*

$$\begin{tikzcd}
Sets^{\mathbb{C}^{op}} \arrow[rr, shift left=3, "u_!"] \arrow[rr, shift right=3, "u_*"'] \arrow[dr, shift left, "\Pi"] \arrow[dr, shift right=3, "\Gamma"'] & & Sets^{\mathbb{D}^{op}} \arrow[ll, "u^*"'] \arrow[dl, shift right, "\Pi"'] \arrow[dl, shift left=3, "\Gamma"] \\
& Sets \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

obtained from $\mathbb{C} \xrightarrow{u} \mathbb{D}$ *is commutative in the sense that*

$$u^* \circ \Delta = \Delta$$
$$\Gamma \circ u_* = \Gamma$$
$$\Pi \circ u_! = \Pi$$

Proof.
The couple $(u^* \circ \Delta, \Gamma \circ u_*)$ is a geometric morphism, as easily verified. The first two identities follow by uniqueness of such morphisms. The last, from uniqueness of adjunctions.

Remark 13.1.4 We defined $u_!$ by the formula

$$u_!X(D) = colim_{\{h_C \to X\}} \mathbb{D}(D, u(C))$$

On the other hand, $u_!$ is usually defined by the left Kan extension

$$Lan_u X(D) = colim_{\{D \to u(C)\}} \mathbb{C}(h_C, X)$$

where things seem to have been turned around. To explain what is going on, we will formulate a 'commutation' lemma that will be useful later on.

Let $\Phi \in Sets^{\mathbb{C}^{op}}$ and $M \in Sets^{\mathbb{C}}$. We define Fig^M to be the category of generic figures of M whose objects are couples (C, σ), where $\sigma \in M(C)$ and a morphism $(C', \sigma') \xrightarrow{f} (C, \sigma)$ is a map $C' \xrightarrow{f} C$ such that $M(f)(\sigma') = \sigma$. Thus we have a forgetful functor $(Fig^M)^{op} \xrightarrow{{U_M}^{op}} \mathbb{C}^{op}$ which sends (C, σ) into C. Recall that we also have a forgetful functor $Fig(\Phi) \xrightarrow{U_\Phi} \mathbb{C}^{op}$. We may consider U_Φ as a functor $(Fig(\Phi))^{op} \xrightarrow{U_\Phi} \mathbb{C}$.

Define

$$\begin{aligned} \Phi[M] &= colim\ \Phi \circ {U_M}^{op} \\ &= colim_{\{h^C \to M\}} \Phi(C) \\ M[\Phi] &= colim\ M \circ U_\Phi \\ &= colim_{\{h_C \to \Phi\}} M(C) \end{aligned}$$

We let $M(C) \xrightarrow{\eta_x} M[\Phi]$ and $\Phi(C) \xrightarrow{\eta_\alpha} \Phi[M]$ be tha canonical maps into the corresponding colimits (with $x \in \Phi(C)$ and $\alpha \in M(C)$.)

Lemma 13.1.5 (Commutation lemma) *There is a canonical bijection*

$$\Phi[M] \xrightarrow{\theta} M[\Phi]$$

such that

$$\begin{tikzcd} \Phi[M] \arrow[rr, "\theta"] & & M[\Phi] \\ & \Phi(C) \times M(C) \arrow[ul, "\mu_C"] \arrow[ur, "\nu_C"'] & \end{tikzcd}$$

is commutative for each C, where $\mu_C(x,\alpha) = \eta_\alpha(x)$ and $\nu_C(x,\alpha) = \eta_x(\alpha)$.

Proof.
Define $\Phi(C) \xrightarrow{\tau_\alpha} M[\Phi]$ by the formula $\tau_\alpha(x) = \eta_x(\alpha)$ and show that for every $(C',\alpha') \xrightarrow{f} (C,\alpha)$ the following diagram is commutative

$$\begin{tikzcd} (C,\alpha) & & \Phi(C) \arrow[dr, "\tau_\alpha"] \arrow[dd, "\Phi(f)"'] & \\ & & & M[\Phi] \\ (C',\alpha') \arrow[uu, "f"] & & \Phi(C') \arrow[ur, "\tau_{\alpha'}"'] & \end{tikzcd}$$

This proves the existence of a unique $\Phi[M] \xrightarrow{\theta} M[\Phi]$ such that $\theta \circ \eta_\alpha = \tau_\alpha$, by the universal property of colimits. Details are left as an exercise. $\square$

To formulate our next result we need the following definition:

A category $\mathbb{I}$ is *filtered* if

(i) $\exists i \in \mathbb{I}$

(ii) $\forall i, j \;\; \exists k$ such that
$$\begin{tikzcd} i \arrow[dr, dashed] & \\ & k \\ j \arrow[ur, dashed] & \end{tikzcd}$$

(iii) $\forall i \overset{u}{\underset{v}{\rightrightarrows}} j \qquad \exists k \; i \overset{u}{\underset{v}{\rightrightarrows}} j \overset{w}{\dashrightarrow} k \qquad$ such that $w \circ u = w \circ v$

A category $\mathbb{I}$ is *cofiltered* iff $\mathbb{I}^{op}$ is filtered.

Remark 13.1.6 *If the category $\mathbb{I}$ is a poset then filtered is the same as directed.*

The importance of the notion of a filtered category comes from

Theorem 13.1.7 (Commutation of limits and filtered colimits) *If $\mathbb{I}$ is a filtered category then*

$$colim : Sets^{\mathbb{I}} \longrightarrow Sets$$

is an exact functor, i.e., it preserves $\mathbb{1}$, pullbacks and equalizers.

Proof.
See [28]

Proposition 13.1.8 *Consider the diagram of adjoint functors*

$$Sets^{\mathbb{C}^{op}} \begin{array}{c} \xrightarrow{u_!} \\ \xleftarrow{u^*} \\ \xrightarrow[u_*]{} \end{array} Sets^{\mathbb{D}^{op}}$$

where $u_! \dashv u^* \dashv u_*$, obtained from a functor $\mathbb{C} \xrightarrow{u} \mathbb{D}$. Then $u_!$ is exact iff for every $D \in \mathbb{D}$ the comma category $D \downarrow u$ is cofiltered.

Proof.
Recall (cf. [28]) that an object of $D \downarrow u$ is a couple (C, α), where $D \xrightarrow{\alpha} u(C)$ is a map of $\mathbb{D}$ and a morphism $(C', \alpha') \xrightarrow{f} (C, \alpha)$ is a map $C' \xrightarrow{f} C$ such that the diagram

$$\begin{tikzcd} D \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & u(C) \\ & u(C') \arrow[u, "u(f)"'] \end{tikzcd}$$

is commutative.

$\leftarrow$: Assume that the comma category $D \downarrow u$ is cofiltered, for every D. By definition $u_!X(D) = colim_{C-->X} h_{u(C)}$. By the commutation lemma (abusing the language) $u_!X(D) = colim_{D \to u(C)} X(C)$. Since the index category $(D \downarrow u)^{op}$ is filtered, the functor colim is exact and this implies the result.

$\rightarrow$: Assume that $u_!$ is exact. We shall prove that $(D \downarrow u)^{op}$ satisfies conditions (i) and (ii) of the definition of a filtered category. Condition (iii) is left as an exercise.
(i) Follows from $1 = u_!\mathbb{1}(D) = colim_{D \to u(C)} 1$. By definition of colimit, there is some $u(C) \longrightarrow D$.
(ii) Assume that

$$\begin{tikzcd} & u(C) \\ D \arrow[ur, "\alpha"] \arrow[dr, "\alpha'"'] & \\ & u(C') \end{tikzcd}$$

From $u_!(h_C \times h_{C'})(D) = u_! h_C(D) \times u_! h_{C'}(D)$ it follows (by definition of $u_!$ and the commutation lemma) that $h_{u(C)}(D) \times h_{u(C')}(D) = colim_{D \longrightarrow u(C'')}(h_C \times h_{C'})(C'')$. Thus, (α, α') belongs to this colimit, i.e., there are $D \xrightarrow{\alpha''} u(C'')$ and

$$\begin{tikzcd} & C \\ C'' \arrow[ur, "f"] \arrow[dr, "f'"'] & \\ & C' \end{tikzcd}$$

such that the diagram

$$\begin{tikzcd} & u(C) & \\ D \arrow[ur, "\alpha"] \arrow[rr, "\alpha''"] \arrow[dr, "\alpha'"'] & & u(C'') \arrow[ul, "u(f)"'] \arrow[dl, "u(f')"] \\ & u(C') & \end{tikzcd}$$

is commutative.

Corollary 13.1.9 *Let $\mathbb{C}$ and $\mathbb{D}$ be categories with finite limits and let $\mathbb{C} \xrightarrow{u} \mathbb{D}$ be an exact functor. Then in the diagram of adjoint functors*

$$\begin{tikzcd} Sets^{\mathbb{C}^{op}} \arrow[r, shift left=2, "u_!"] \arrow[r, shift right=2, "u_*"'] & Sets^{\mathbb{D}^{op}} \arrow[l, "u^*"'] \end{tikzcd}$$

where $u_! \dashv u^* \dashv u_*$, $u_!$ is also an exact functor.

Remark 13.1.10 *The pair* (Δ, Γ) *such that* $\Pi \dashv \Delta \dashv \Gamma$ *is obtained as a particular case when* $\mathbb{D} = \mathbb{1} : \Delta = u^*$, $\Pi = u_!$ *and* $\Gamma = u_*$.

Corollary 13.1.11 *The functor* $\Pi : Sets^{\mathbb{C}^{op}} \longrightarrow Sets$ *is exact iff the category* $\mathbb{C}$ *is co-filtered.*

Proof.
For the case under consideration, the condition that $D \downarrow u$ is cofiltered for all D boils down to $\mathbb{C}$ is cofiltered. Notice that the 'if' direction is identical to the basic statement that filtered colimits commute with finite limits (see theorem 13.1.7).

Corollary 13.1.12 *If* $\Pi\Omega = 1$, *then* Π *is not exact.*

Proof.
Immediate from Lawvere's theorem 12.2.3 and the following

Lemma 13.1.13 *A cofiltered category satisfies the Ore condition*

Proof (of lemma).
Left as an exercise.

Notice that when $\mathbb{C} = V \xrightarrow{v} L$ and $\mathbb{D} = \mathbb{1}$, $\mathbb{C}$ and $\mathbb{D}$ have finite limits and $\mathbb{C} \xrightarrow{u} \mathbb{D}$ obviously preserves them. So, in the case:

$$\begin{tikzcd} Bouquets \arrow[r, shift left=2, "u_!=\Pi"] \arrow[r, shift right=2, "u_*=\Gamma"'] & Sets \arrow[l, "u^*=\Delta"'] \end{tikzcd}$$

we have that Π is exact.

13.2 The right adjoint to u_*

Let $\mathbb{C} \xrightarrow{u} \mathbb{D}$ be a functor. We know that u gives rise to a diagram

$$\begin{tikzcd} Sets^{\mathbb{C}^{op}} \arrow[r, shift left=2, "u_!"] \arrow[r, shift right=2, "u_*"'] & Sets^{\mathbb{D}^{op}} \arrow[l, "u^*"'] \end{tikzcd}$$

with $u_! \dashv u^* \dashv u_*$. The aim of this section is to generalize Lawvere's theorem 11.5.1 by giving necessary and sufficient conditions for the existence of a right adjoint to u_*. Recall that $u^*Y = Y \circ u$.

Theorem 13.2.1 *The functor u_* has a right adjoint B iff for every $D \in \mathbb{D}$ the functor $h_D \circ u \in \overline{\mathbb{C}}$, where $\overline{\mathbb{C}}$ is the Cauchy completion of $\mathbb{C}$.*

Proof.
Recalling the definition of Cauchy completion of a category (see section 5.2) the condition for the existence of B can be reformulated as: for all D there is a diagram

$$u^* h_D \underset{\mu}{\overset{\nu}{\rightleftarrows}} h_C$$

with $\nu\mu = 1$.

Before proceeding with the proof, we need some preliminaries. A $\mathbb{C}$-set X is *continuous* iff for every small category $\mathbb{I}$ and every functor $\mathbb{I} \xrightarrow{F} Sets^{\mathbb{C}^{op}}$

$$[X, colim_i F_i] = colim_i [X, F_i]$$

Lemma 13.2.2 *Let $\mathbb{C}$ be a category and $X : \mathbb{C} \longrightarrow Sets$. Then X is continuous iff $X \in \overline{\mathbb{C}}$.*

Proof.
$\rightarrow$: Assume the existence of B. Since u_* has a right adjoint, it preserves colimits. The following computation shows that for every $D \in \mathbb{D}$, $h_D \circ u$ is continuous:

$$\begin{aligned} [h_D \circ u, colim_i F_i] &= [h_D, u_* colim_i F_i] \\ &= [h_D, colim_i u_* F_i] \\ &= colim_i [h_D, u_* F_i] \\ &= colim_i [h_D \circ u, F_i] \end{aligned}$$

The result follows from the lemma.

$\leftarrow$: Assume that for every $D \in \mathbb{D}$, $h_D \circ u \in \overline{\mathbb{C}}$. We notice that the definition of B is forced. Indeed, BY is the $\mathbb{C}$-set whose generic figures and action by

a change of figure are given :

$$\frac{C \overset{\sigma}{\dashrightarrow} BY}{u_*h_C \xrightarrow{\sigma} Y} \qquad \frac{\begin{tikzcd} & C \arrow[r, dashed, "\sigma"] & BY \\ C' \arrow[ur, "f"] \arrow[urr, dashed, "\sigma.f"'] & & \end{tikzcd}}{\begin{tikzcd} & u_*h_C \arrow[r, "\sigma"] & Y \\ u_*h_{C'} \arrow[ur, "u_*h_f"] \arrow[urr, "\sigma\circ u_*h_f"'] & & \end{tikzcd}}$$

Before proving the result, we introduce a piece of notation: if $F \dashv G$, we let $\overline{(\;)}$ to be the map that sends a morphism into its transpose. Thus:

$$\frac{FB \xrightarrow{f} A}{B \xrightarrow{\overline{f}} GA} \quad \frac{B \xrightarrow{g} GA}{FB \xrightarrow{\overline{g}} A}$$

Notice that $\overline{\overline{f}} = f$ and $\overline{\overline{g}} = g$.

The unit of the adjunction is the morphism

$$X \xrightarrow{\eta_X} Bu_*X$$

whose value on a C-figure ξ of X is the C-figure $u_*h_C \xrightarrow{u_*(\xi)} u_*X$. Thus,

$$\eta_X(\xi) = u_*(\xi)$$

It is clear and easy to see that η is a natural transformation.

The counit of the adjunction is the morphism

$$u_*BY \xrightarrow{\epsilon_Y} Y$$

defined as follows: take $D \in \mathbb{D}$. By hypothesis, there is a diagram

$$\begin{tikzcd} u^*h_D \arrow[r, hook, shift right, "\mu"'] & h_C \arrow[l, shift right, "\nu"'] \end{tikzcd}$$

with $\nu\mu = 1$. We let
$$\epsilon_{_Y}(\tau) = \tau(\nu) \circ \overline{\mu}$$
where τ is a D-figure of u_*BY. Let us show that this definition does not depend on the choice of C, μ, ν.

Assume that u^*h_D is a retract of both h_C and $h_{C'}$:

$$\begin{tikzcd} u^*h_D \arrow[r, hook, shift right, "\mu"'] & h_C \arrow[l, two heads, shift right, "\nu"'] \end{tikzcd} \qquad \begin{tikzcd} u^*h_D \arrow[r, hook, shift right, "\mu'"'] & h_{C'} \arrow[l, two heads, shift right, "\nu'"'] \end{tikzcd}$$

with $\nu\mu = 1$ and $\nu'\mu' = 1$.

From the commutativity of the diagram

$$\begin{tikzcd} & h_C \\ u^*h_D \arrow[ur, "\mu"] \arrow[dr, "\mu'"'] & \\ & h_{C'} \arrow[uu, "\mu\nu'"'] \end{tikzcd}$$

it follows (by adjointness) that the following diagram is commutative

$$\begin{tikzcd} & u_*h_C \\ h_D \arrow[ur, "\overline{\mu}"] \arrow[dr, "\overline{\mu'}"'] & \\ & u_*h_{C'} \arrow[uu, "u_*(\mu\nu')"'] \end{tikzcd} \qquad \star$$

On the other hand, from the commutativity of

$$\begin{tikzcd} h_C \arrow[dr, "\nu"] & \\ & u^*h_D \\ h_{C'} \arrow[uu, "u_*(\mu\nu')"] \arrow[ur, "\nu'"'] & \end{tikzcd}$$

it follows (from the definition of the action of B) the commutativity of

$$\begin{tikzcd}
u_*h_C \arrow[dr, "\tau(\nu)"] & \\
& Y \\
u_*h_{C'} \arrow[uu, "u_*(\mu\nu')"] \arrow[ur, "\tau(\nu')"'] &
\end{tikzcd} \qquad \star\star$$

From $(\star)$ and $(\star\star)$, we conclude that $\tau(\nu) \circ \overline{\mu} = \tau(\nu') \circ \overline{\mu'}$, showing that ϵ is well defined.

It is not so clear, but still true that ϵ is a natural transformation: take

$$\begin{tikzcd}
u^*h_D \arrow[r, hook, shift right, "\mu"'] & h_C \arrow[l, two heads, shift right, "\nu"']
\end{tikzcd} \qquad \begin{tikzcd}
u^*h_D \arrow[r, hook, shift right, "\mu'"'] & h_{C'} \arrow[l, two heads, shift right, "\nu'"']
\end{tikzcd}$$

and $D' \xrightarrow{q} D$. We have to show that the diagram

$$\begin{tikzcd}
u_*BY(D) \arrow[r, hook, "(\epsilon_Y)_D"] \arrow[d, "u_*BY(g)"'] & Y(D) \arrow[d, "Y(g)"] \\
u_*BY(D') \arrow[r, "(\epsilon_Y)'_D"'] & Y(D')
\end{tikzcd}$$

is commutative, i.e., if $\tau \in u_*BY(D)$, $(\tau(\nu) \circ \overline{\mu}).g = \tau(u^*(g) \circ \nu') \circ \overline{\mu'}$. But from the commutativity of

$$\begin{tikzcd}
u^*h_D \arrow[r, "\mu"] & h_C \arrow[dr, "\tau\nu"] & \\
& & BY \\
u^*h_{D'} \arrow[uu, "u^*(g)"] \arrow[r, "\mu'"'] & h_{C'} \arrow[uu, "\mu u^*(g)\nu'"'] \arrow[ur, "\tau u^*(g)\nu'"'] &
\end{tikzcd}$$

it follows the commutativity of

$$\begin{tikzcd}
h_D \arrow[r, "\overline{\mu}"] & u_*h_C \arrow[dr, "\tau(\nu)"] & \\
 & & Y \\
h_{D'} \arrow[uu, "g"] \arrow[r, "\overline{\mu'}"'] & u_*h_{C'} \arrow[uu] \arrow[ur, "\tau(u^*(g)\nu')"'] &
\end{tikzcd}$$

Hence $\tau(\nu) \circ \overline{\mu} \circ g = \tau(u^*(g)\nu')\overline{\mu'}$.

In the proof we need to compute $u_*X \xrightarrow{u_*(\eta_X)} u_*Bu_*X$. The computation is carried out as follows:

$$\begin{array}{c}
D \dashrightarrow^{\theta} u_*X \xrightarrow{\eta_X} Bu_*X \\
\hline
u^*h_D \xrightarrow{\theta} X \xrightarrow{\eta_X} Bu_*X \\
\hline
u^*h_D \xrightarrow{\eta_X \circ \theta} Bu_*X \\
\hline
D \overset{\eta_X \circ \theta}{\dashrightarrow} u_*Bu_*X
\end{array}$$

Thus,

$$(u_*\eta_X)(\theta) = \eta_X \circ \theta$$

To conclude the proof of the theorem it is enough (because of proposition 8.1.3) to check the formulas $\begin{cases} \epsilon_{u_*X} \circ u_*\eta_X = 1_{u_*X} \\ B\epsilon_Y \circ \eta_{BY} = 1_{BY} \end{cases}$

Let us prove the first, the other is left in exercise.

We have to compute the composition of the following morphisms:

$$u_*X \xrightarrow{u_*\eta_X} u_*Bu_*X \xrightarrow{\epsilon_{u_*X}} u_*X$$

From the previous computation and the definitions of ϵ and η we obtain

$$\begin{aligned}
(\epsilon_{u_*X} \circ u_*\eta_X)(\theta) &= \epsilon_{u_*X}((u_*\eta_X)(\theta)) \\
&= \epsilon_{u_*X}(\eta_X \circ \theta) \\
&= (\eta_X \circ \theta)(\nu) \circ \overline{\mu}
\end{aligned}$$

where θ is a D-figure of u_*X, i.e., a morphism $u^*h_D \xrightarrow{\theta} X$. We claim that

$$(\eta_X \circ \theta)(\nu) \circ \overline{\mu} = \theta$$

We have the following equivalences

$$
\begin{tikzcd}
 & u_*(h_C) \arrow[r, "\eta_X(\theta(\nu))"] & u_*X \\
h_D \arrow[ur, "\overline{\mu}"] \arrow[urr, "\eta_X(\theta(\nu))\circ\overline{\mu}"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & u_*(h_C) \arrow[r, "u_*(\theta(\nu))"] & u_*X \\
h_D \arrow[ur, "\overline{\mu}"] \arrow[urr, "u_*(\theta(\nu))\circ\overline{\mu}"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & h_C \arrow[r, "\theta(\nu)"] & X \\
u^*h_D \arrow[ur, "\mu"] \arrow[urr, "\theta(\nu)\circ\mu"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & h_C \arrow[r, "\theta\circ\nu"] & X \\
u^*h_D \arrow[ur, "\mu"] \arrow[urr, "\theta"'] & &
\end{tikzcd}
$$

Thus, $\eta_X(\theta(\nu)) \circ \overline{\mu} = \theta$. $\Box$

Remark 13.2.3 As for the other functors, when all the functors B exist they commute in the sense that the diagram

$$
\begin{tikzcd}
Sets^{\mathbb{C}^{op}} & & Sets^{\mathbb{D}^{op}} \arrow[ll, "B"'] \\
 & Sets \arrow[ul, "B"] \arrow[ur, "B"'] &
\end{tikzcd}
$$

is commutative

Example: the functor B between bouquets and special graphs

Let $\mathbb{C}$ be the category

$$
\begin{tikzcd}
V \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A \arrow[loop above, "\sigma"] \arrow[loop below, "\tau"]
\end{tikzcd}
$$

with the relations

$$\begin{cases} \sigma^2 = \sigma\tau = \sigma \\ \tau^2 = \tau\sigma = \tau \\ \sigma s = \sigma t = s \\ \tau s = \tau t = t \end{cases}$$

To represent graphically a $\mathbb{C}$-set, recall the category

$$\begin{tikzcd} V \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & A \arrow[l, shift left=3, "l"] \arrow[loop above, "\sigma"] \arrow[loop below, "\tau"] \end{tikzcd}$$

which generates the reflexive graphs.

Thinking again of V as *the vertex* and A as *the arrow*, the maps s and t are extractions of vertices from the arrow (there are two), whereas σ and τ are extractions of loops from the arrow: the first whose source (and target) is the source of A and the second whose source (and target) is the target of A. Contrary to the case of reflexive graphs, we cannot extract arrows from the vertices (i.e., the map l is missing).

Thus, a graphical representation of a $\mathbb{C}$-set is a graph such that every arrow has two distinguished loops: $\sigma(f)$ whose source (and target) is the source of f and $\tau(f)$ whose source (and target) is the target of f. Unlike reflexive graphs, there may be vertices without loops.

Let $\mathbb{D}$ be the category that generates *Bouquets* :

$$V \xrightarrow{v} L$$

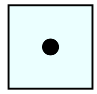
Define the functor $\mathbb{C} \xrightarrow{u} \mathbb{D}$ as follows:

$$\begin{cases} uV = V \ , \ uA = L \\ u(s) = u(t) = v \\ u(\sigma) = u(\tau) = 1_L \end{cases}$$

We claim that B exists. By theorem 13.2.1 it is enough to show that $h_V \circ u$ and $h_L \circ u$ belong to the Cauchy completion of $\mathbb{C}$ or, what it amounts to the same, are retracts of representables of $Sets^{\mathbb{C}^{op}}$. As done on previous occasions we will compute the representables h_V and h_A and represent them graphically. Let us extract the generic figures of h_V

$$\frac{\dfrac{V \dashrightarrow h_V}{V \longrightarrow V}}{1_V} \qquad \frac{\dfrac{A \dashrightarrow h_V}{A \longrightarrow V}}{\emptyset}$$

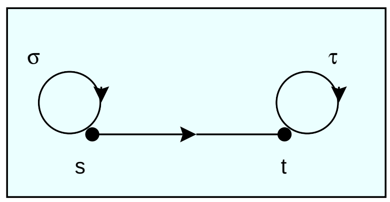
with no non trivial incidence relations.Thus

$h_V =$ 

Let us do the same with h_A.

$$\begin{array}{c} V \dashrightarrow h_A \\ \hline V \longrightarrow A \\ \hline s, t \end{array} \qquad \begin{array}{c} A \dashrightarrow h_A \\ \hline A \longrightarrow A \\ \hline \sigma, \tau, 1_A \end{array}$$

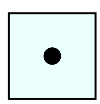
with incidence relations $\sigma s = \sigma t = s$, $\tau s = \tau t = t$, $1_A s = s$, $1_A t = t$. Therefore

$h_A =$ 

We proceed to compute and represent graphically $h_V \circ u$ and $h_L \circ u$ by extracting their generic figures

$$\begin{array}{c} V \dashrightarrow h_V \circ u \\ \hline uV \longrightarrow V \\ \hline V \longrightarrow V \\ \hline 1_V \end{array} \qquad \begin{array}{c} A \dashrightarrow h_V \circ u \\ \hline uA \longrightarrow V \\ \hline L \longrightarrow V \\ \hline \emptyset \end{array}$$

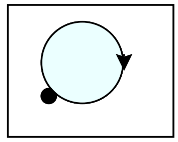
Thus,

$h_V \circ u =$ 

Similarly,

$$\begin{array}{c} V \dashrightarrow h_L \circ u \\ \hline uV \longrightarrow L \\ \hline V \longrightarrow L \\ \hline v \end{array} \qquad \begin{array}{c} A \dashrightarrow h_L \circ u \\ \hline uA \longrightarrow L \\ \hline L \longrightarrow L \\ \hline 1_L \end{array}$$

Thus, $h_L \circ u = \mathbb{1}$, i.e.,

$h_L \circ u =$ 

Clearly $h_V \circ u = h_V$ and $h_L \circ u$ is a retract of h_A and this proves the existence of B. To compute B we need the following result, safely left to the reader:

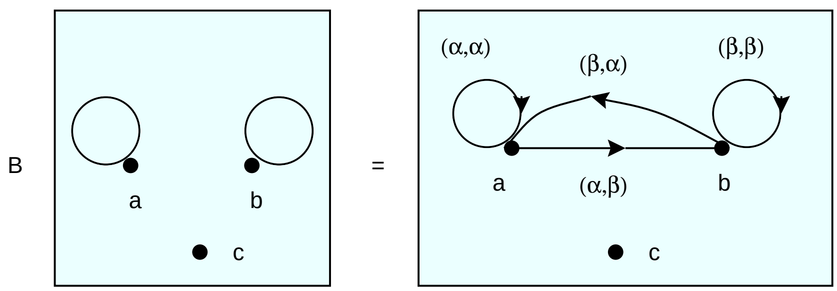
$$u_* h_V = h_V \quad u_* A = \mathbb{1} + \mathbb{1}$$

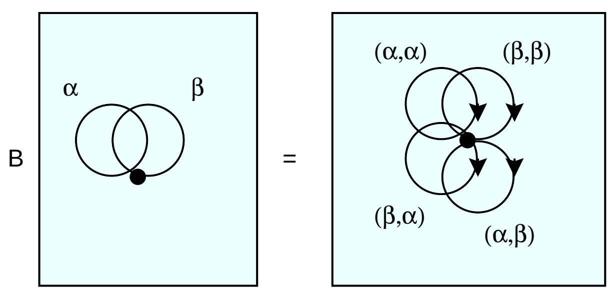
With these preliminaries out of the way, BX may be computed as usual (where X is a bouquet)

$$\begin{array}{c} V \dashrightarrow BX \\ \hline u_* V \longrightarrow X \\ \hline V \longrightarrow X \\ \hline \text{vertices of } X \end{array} \qquad \begin{array}{c} A \dashrightarrow BX \\ \hline u_* A \longrightarrow L \\ \hline \mathbb{1} + \mathbb{1} \longrightarrow X \\ \hline \text{couples of loops of } X \end{array}$$

with the obvious incidence relations (left as exercise).

As examples:





Notice that this is quite similar to the functor $B : Sets \longrightarrow Rgraphs$. Indeed, if S is a set, then BS is the reflexive graph obtained by putting an arrow whenever possible. In the present case, recall that any arrow must have a loop at each end: this explains why we can put arrows only between loops.

13.3 Comparing Ω objects

Return to a functor $\mathbb{C} \xrightarrow{u} \mathbb{D}$ between two small categories. As we saw, this functor gives rise to a diagram:

$$\begin{tikzcd} Sets^{\mathbb{D}^{op}} \arrow[r, shift left=2, "u_!"] & Sets^{\mathbb{C}^{op}} \arrow[l, "u^*" description] \arrow[l, phantom] \\ \end{tikzcd}$$

$$Sets^{\mathbb{D}^{op}} \; \overset{u_!}{\underset{u_*}{\rightrightarrows}} \!\!\!\! \overset{u^*}{\longleftarrow} \; Sets^{\mathbb{C}^{op}}$$

Our question is whether Ω objects in these categories of presheaves are comparable. The answer is given by the following

Proposition 13.3.1 *The functors $u_! \dashv u^* \dashv u_*$ induce a diagram*

$$u_*\Omega_{\mathbb{C}} \; \overset{\lambda}{\underset{\gamma}{\rightrightarrows}} \!\!\!\! \overset{\delta}{\longleftarrow} \; \Omega_{\mathbb{D}}$$

of internally adjoint morphisms $\lambda \dashv \delta \dashv \gamma$.

Proof.
The definition of δ proceeds as follows

$$\begin{array}{c} h_D \xrightarrow{p} \Omega_{\mathbb{D}} \\ \hline p \hookrightarrow h_D \\ \hline u^*p \hookrightarrow u^*h_D \\ \hline u^*h_D \longrightarrow \Omega_{\mathbb{C}} \\ \hline h_D \xrightarrow{\delta(p)} u_*\Omega_{\mathbb{C}} \end{array} \qquad \begin{array}{c} \downarrow \ \ \text{(applying } u^*) \\ (\Omega_{\mathbb{C}} \ \ \text{classifies}) \\ (u^* \dashv u_*) \end{array}$$

As for the definition of γ :

$$\begin{array}{c} h_D \xrightarrow{K} u_*\Omega_{\mathbb{C}} \\ \hline u^*h_D \longrightarrow \Omega_{\mathbb{C}} \\ \hline K \hookrightarrow u^*h_D \\ \hline \end{array} \qquad \downarrow \ \ \text{(apply } u_*\text{, take pullback)}$$

$$\begin{tikzcd} u_*K \arrow[r, hook] & u_*u^*h_D \\ p \arrow[u] \arrow[r, hook] & h_D \arrow[u, "\eta_D"'] \end{tikzcd}$$

$$\begin{array}{c} \hline h_D \xrightarrow{\gamma(K)} \Omega_{\mathbb{D}} \end{array} \qquad \downarrow \ \ (\Omega_{\mathbb{D}} \ \ \text{classifies})$$

Finally let us define λ as follows:

$$
\begin{array}{cl}
h_D \xrightarrow{K} u_*\Omega_{\mathbb{C}} & \\ \hline
u^*h_D \longrightarrow \Omega_{\mathbb{C}} & \\ \hline
K \hookrightarrow u^*h_D & \\ \hline
& \downarrow \quad \text{(apply } u_!\text{, take image)} \\
\begin{array}{ccc} u_!K & \hookrightarrow & u_!u^*h_D \\ \big\downarrow & & \big\downarrow{\scriptstyle \epsilon_D} \\ p & \hookrightarrow & h_D \end{array} & \\ \hline
h_D \xrightarrow{\lambda(K)} \Omega_{\mathbb{D}} & \downarrow \quad (\Omega_{\mathbb{D}} \ \text{ classifies})
\end{array}
$$

We leave as an exercise to check that

$$\lambda \dashv \delta \dashv \gamma$$

Furthermore, one can show that $u_*\Omega_{\mathbb{C}}$ is an internal frame and that δ is the only internal frame morphism between these internal frames. This is obvious by using the internal language mentioned in remark 9.4.1, but a long exercise if proved from scratch.

We say that the (essential) geometric morphism $u^* \dashv u_*$ is *open* if the Frobenius condition

$$\lambda(\delta a \wedge b) = a \wedge \lambda b$$

holds, where a is C-figure of $\Omega_{\mathbb{D}}$ and b a C-figure of $u_*\Omega_{\mathbb{C}}$.

Remark 13.3.2 From the logical point of view, openness means that 'u^* preserves infinitary full first-order logic' in the sense that it preserves universal quantifiers ($\forall_\Phi$), implications ($\rightarrow$) and negations ($\neg$) as well as existential quantifiers ($\exists_\Phi$), arbitrary sups ($\bigvee$) and arbitrary infs ($\bigwedge$) which are preserved by any (not necessarily open) essential u^*.

Example: graphs and bisets

We saw that $2 \xrightarrow{u} (V \underset{t}{\overset{s}{\rightrightarrows}} A)$ gives rise to a diagram

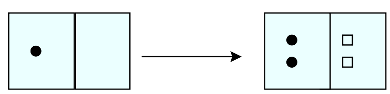
$$
\begin{tikzcd}
Graphs \arrow[r, shift left=2, "u_!"] \arrow[r, shift right=2, "u_*"'] & Bisets \arrow[l, "u^*"']
\end{tikzcd}
$$

We shall compute the morphisms in the diagram derived from the previous one

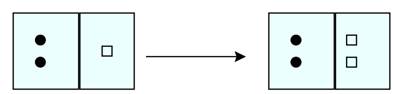
$$\Omega_2 \; \underset{\gamma}{\overset{\lambda}{\underset{\xleftarrow{\delta}}{\rightarrow}}} \; \Omega_G$$

We first describe $u_*\Omega_2$ as a graph and as an internal poset by extracting its generic figures.

$$\frac{\dfrac{V \dashrightarrow u_*\Omega_2}{h_V \longrightarrow u_*\Omega_2}}{u^*h_V \longrightarrow \Omega_2} \qquad \frac{\dfrac{A \dashrightarrow u_*\Omega_2}{h_A \longrightarrow u_*\Omega_2}}{u^*h_A \longrightarrow \Omega_2}$$

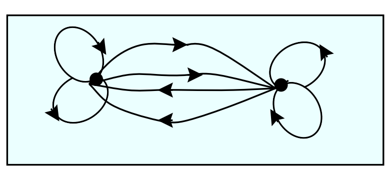


2 vertices

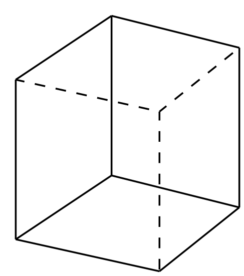


8 arrows

Taking into account the incidence relations and the order between its generic figures we obtain the following representation for $u_*\Omega_2$ as a graph and as posets of vertices and arrows given by Hasse diagrams:

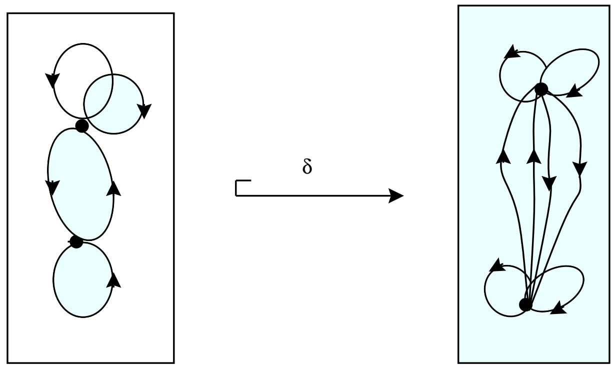


vertices

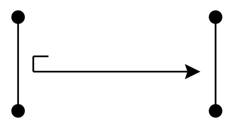


arrows

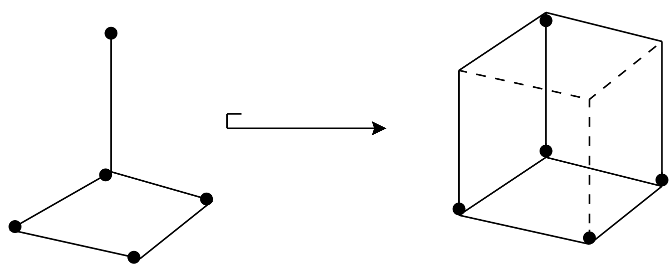
The internal functor



is the graph morphism which embeds, at the level of posets



and



The internal functors λ and γ have the following simple descriptions:

λ sends a vertex of the cube into the nearest upper point $\bullet$ (identified with the corresponding point of the domain)

γ sends a vertex of the cube into the nearest lower point $\bullet$ (identified with the corresponding point of the domain)

Notice that $u_*\Omega_2$ is an internal Boolean algebra in the sense that pointwise it is a Boolean algebra. This implies that the geometric morphism $u^* \dashv u_*$ is *not* open. In fact, the Frobenius condition is equivalent to the condition that δ preserves the implication $\rightarrow$. But δ does not preserve implications, since it would preserve negations and this, in turn, would imply that for every a, $\delta(a \vee \neg a) = 1$. Since δ is injective, this would mean that for every a, $(a \vee \neg a) = 1$. In other words, the domain would be a Boolean algebra.

WINDOW 13.3.1

A categorical approach to modal operators

Since most logical operations may be defined as maps from a power of Ω into Ω, it is natural to view modal operators as endo maps of Ω. The following proposition (see exercise 9.2.1) seems to end any hopes of carrying out this program to fruition

Proposition 13.3.3 *Assume that* $\Box : \Omega \longrightarrow \Omega$ *is a morphism such that*

(1) $\Box \leq 1_\Omega$

(2) $\Box \top = \top$

Then $\Box = 1_\Omega$.

One way out of this dilemma was already known to Aristotle: change is always relative to a standard of rest. Instead of one category of $\mathbb{C}$-sets, we should consider two: one to provide the standard of rest and the other to act as theater of change (and modalities).

Returning to the general situation $u : \mathbb{C} \longrightarrow \mathbb{D}$ we consider $Sets^{\mathbb{D}^{op}}$ as providing the standard of rest and $Sets^{\mathbb{C}^{op}}$ as the theater of change (and modalities). We have of course an essential geometric morphism $u_! \dashv u_* \dashv u_*$, which induces adjoint internal functors $\lambda \dashv \delta \dashv \gamma$, where

$$\delta : \Omega_{\mathbb{D}} \longrightarrow u_* \Omega_{\mathbb{C}}$$

In terms of these functors we can define two operators

$$\Diamond : u_* \Omega_{\mathbb{C}} \longrightarrow u_* \Omega_{\mathbb{C}}$$

$$\Box : u_* \Omega_{\mathbb{C}} \longrightarrow u_* \Omega_{\mathbb{C}}$$

by $\Diamond = \delta\lambda$ and $\Box = \delta\gamma$. These operators have the following properties:

$$\Box \leq Id \leq \Diamond$$
$$\Box^2 = \Box, \quad \Diamond^2 = \Diamond$$
$$\Diamond \dashv \Box$$

This is the starting point of a topos-theoretic theory of modal operators. The interested reader can find details in the following references: [21], [39], [42], [40] and [35] .

Another approach to modal operators is to give up their naturality and, consequently, give up their representability as endo maps of Ω. Recall that co-Heyting logical operations are not natural either, except in degenerate (i.e. Boolean) cases. Indeed, modal operators may be defined in terms of bi-Heyting operations. For the resulting theory, the reader may consult [41].

EXERCISE 13.3.1

(1) Complete the details of this example

(2) Compute λ, δ and γ in the other examples of geometric morphisms described in this chapter. Which, if any, is open?

(3) Show that $\Diamond$ and $\Box$ have the properties stated in the remark.

14 Points of a category of $\mathbb{C}$-sets

A *point* of the category $Sets^{\mathbb{C}^{op}}$ is a geometric morphism $Sets \xrightarrow{p} Sets^{\mathbb{C}^{op}}$. (This terminology is justified since $Sets$ is a kind of terminal object in the category of presheaves, as we saw in proposition 11.1.1).

Unraveling this definition, this means that a point p of $Sets^{\mathbb{C}^{op}}$ is a couple $p = (p^*, p_*)$

$$Sets \underset{p_*}{\overset{p^*}{\leftrightarrows}} Sets^{\mathbb{C}^{op}}$$

such that $p^* \dashv p_*$ and p^* preserves $\mathbb{1}$ and pullbacks.

A functor is *exact* if it preserves $\mathbb{1}$ and pullbacks, i.e., finite limits.

Given two points p and q, a *morphism* $p \longrightarrow q$ is, by definition, a natural transformation $p^* \longrightarrow q^*$.

Notice that the natural transformations between p^* and q^* are in bijective correspondence with those between q_* and p_*. Thus, we have the following bijections

$$\begin{array}{c} p \longrightarrow q \\ \hline p^* \longrightarrow q^* \\ \hline q_* \longrightarrow p_* \end{array}$$

With this notion of morphism, the points of $Sets^{\mathbb{C}^{op}}$ form a category: $Pts(Sets^{\mathbb{C}^{op}})$

Examples of points.

(1) The points of *Sets*.

$$Sets \underset{p_*}{\overset{p^*}{\rightleftarrows}} Sets$$

p^* is exact and we have

$$p^*(S) = p^*(\coprod_{s \in S} 1) = \coprod_{s \in S} p^* 1 = \coprod_{s \in S} 1 = S$$

So, $p^* = p_* = Id$ and $Pts(Sets) = \mathbb{1}$

(2) The points of *Bouquets*.
$p^* L = 1$, since $L = 1$ and p^* preserves 1. In the category $V \xrightarrow{v} L$: we have that the (first) projection induces canonical isomorphisms $V \times V = V$ and $L \times L = L$. From the first we obtain, since p^* preserves products, $p^* V \times p^* V = p^* V$ which in turn implies that $p^* V = \emptyset$ or $p^* V = 1$. It seems reasonable to conjecture that there are two points and a morphism between them:

$$\begin{array}{ccc} \emptyset \longrightarrow \mathbb{1} & = & \bullet \\ & & \downarrow \\ \mathbb{1} \longrightarrow \mathbb{1} & = & \bullet \end{array}$$

Thus, our conjecture is: $Pts(Bouquets) = \bullet \longrightarrow \bullet$

(3) The points of $Graphs$.

We first make the following remark. In the category $V \underset{t}{\overset{s}{\rightrightarrows}} A$, $A \longrightarrow \mathbb{1}$ is an epimorphism and this implies that $p^*A \neq \emptyset$. More generally:

Lemma 14.0.4 *Let $A \in Obj(\mathbb{C})$. If $A \longrightarrow \mathbb{1}$ is an epimorphism then $p^*A \neq \emptyset$*

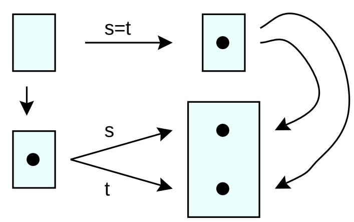
Proof.
See exercise 4.1.1

Let us now consider the following products: $V \times V = V$, $V \times A = 2V$ and $A \times A = A + 2V$. From these, we can deduce that $p^*(V \times V) = p^*V \times p^*V$ which in turn implies that $p^*V = \emptyset$ or $p^*V = \mathbb{1}$. If $p^*V = \emptyset$ then $p^*A = \mathbb{1}$, if $p^*V = \mathbb{1}$ then $p^*A = \mathbb{2}$. So we conjecture that there are two points and two morphisms between them:

$$\emptyset \xrightarrow{s=t} \mathbb{1}$$

$$\mathbb{1} \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{2}$$

and more graphically



Thus, we conjecture that $Pts(Graphs) = \bullet \rightrightarrows \bullet$

In order to analyze the points of the other examples and prove these conjectures we must develop more mathematical tools. Let us start with a definition.

A functor $\mathbb{C} \xrightarrow{M} Sets$ is *filtering* iff the category $Fig(M)$ is filtered. This can be reformulated as follows:

(i) $\exists A \ M(A) \neq \emptyset$

(ii) $\forall A, B$ if $a \in M(A)$ and $b \in M(B)$ then

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

and $\exists c \in M(C)$ such that $M(f)(c) = a$ and $M(g)(c) = b$

(iii) $\forall A \underset{v}{\overset{u}{\rightrightarrows}} B$ if $a \in M(A)$ and $M(u)(a) = M(v)(a)$ then there is a diagram

$$C \xrightarrow{w} A \underset{v}{\overset{u}{\rightrightarrows}} B$$

such that $uw = vw$ and $\exists c \in M(C)$ such that $M(w)(c) = a$.

EXERCISE 14.0.2

(1) Let $\mathbb{C} \xrightarrow{M} Sets$ be a functor and assume that $\mathbb{C}$ is finitely complete (i.e., it has finite limits). Show that M is exact iff M is filtering.

(2) Let $\mathbb{C} \xrightarrow{M} Sets$ be a filtering functor and let $(B, b) \in Fig(M)$. Define

$$(B, b)/Fig(M) \xrightarrow{U} \mathbb{C}^{op}$$

to be the obvious forgetful functor such that $U((B, b) \longrightarrow (A, a)) = A$. Show that $M = colim Y \circ U$, where Y is the Yoneda functor $\mathbb{C}^{op} \xrightarrow{Y} Sets^{\mathbb{C}}$.

Theorem 14.0.5 *Let $\mathbb{C} \xrightarrow{Y} Sets^{\mathbb{C}^{op}}$ where Y is the Yoneda functor, then the functor $Y^* = (\ ) \circ Y$ induces an equivalence of categories:*

$$Y^* : Points(Sets^{\mathbb{C}^{op}}) \xrightarrow{\sim} Filt(\mathbb{C}, Sets)$$

Proof.
Let us first show that $p^* \circ Y$ is a filtering functor, where $p = (p^*, p_*)$ is a point. As an example, let us check condition (ii): assume that $a \in (p^* \circ Y)(A)$ and $b \in (p^* \circ Y)(B)$. Then

$$\begin{aligned}
p^*(h_A) \times p^*(h_B) &= p^*(h_A \times h_B) \\
&= colim_{C -->h_A \times h_B} p^* h_C \\
&= colim_{C \begin{smallmatrix}\nearrow A \\ \searrow B\end{smallmatrix}} p^* h_C \\
&= colim_{C \begin{smallmatrix}\nearrow A \\ \searrow B\end{smallmatrix}} p^* \circ Y(C)
\end{aligned}$$

By definition of colimits in $Sets$, there is $C\begin{smallmatrix} \overset{f}{\nearrow} A \\ \underset{g}{\searrow} B \end{smallmatrix}$ and $z \in (p^* \circ Y)(C)$ such that $z \mapsto x$ via the map $(p^* \circ Y)(C) \longrightarrow (p^* \circ Y)(A)$ and $z \mapsto y$ via the map $(p^* \circ Y)(C) \longrightarrow (p^* \circ Y)(B)$. Thus, condition (ii) is satisfied. The others are left in exercise.

To finish the proof, we show that Y^* is an equivalence, i.e., Y^* is faithful, full and essentially surjective.

Let $p^* \xrightarrow{\alpha} q^*$ and $p^* \xrightarrow{\beta} q^*$ be two different natural tranformations. So, there is $X \in Sets^{\mathbb{C}^{op}}$ and $x \in p^*X$ such that $\alpha_X(x) \neq \beta_X(x)$. Since $p^*X = colim_{h_C-->X} p^*(h_C)$, there is some $h_C \longrightarrow X$ and some $\xi \in p^*(h_C)$ which maps to x under the map $p^*(h_C) \longrightarrow p^*(X)$. Clearly $\alpha_{h_C}(\xi) \neq \beta_{h_C}(\xi)$ and this shows that Y^* is faithful.

To show that Y^* is full, let $p^* \circ Y \xrightarrow{f} q^* \circ Y$ be a natural transformation. For each $X \in Sets^{\mathbb{C}^{op}}$, we define $p^*(X) \xrightarrow{F_X} q^*X$ using the universal property of colimit:

$$\begin{tikzcd}
p^* \circ Y(C) \arrow[r, "\alpha_C"] \arrow[d] & q^* \circ Y(C) \arrow[d] \\
p^*(X) \arrow[r, "\alpha_X"'] & q^*(X)
\end{tikzcd}$$

We have to show that F is a natural transformation, ie., given a natural transformation $Z \xrightarrow{\Phi} X$ $q^*(\Phi) \circ F_Z = F_X \circ p^*(\Phi)$

Let $z \in p^*(Z)$. Proceeding as in the proof of faithfulness, there is some $h_D \longrightarrow p^*(Z)$ and some $\zeta \in p^*(Z)$ which maps to z under the map $p^*(h_D) \longrightarrow p^*(h_D)$. By composing Φ with the map $h_D \longrightarrow Z$ and using $X = colim_{C-->X} h_C$, it follows that the composite $h_D \longrightarrow colim_{C-->X} h_C$ factors through some $h_C \longrightarrow X$. We obtain the following diagrams:

$$\begin{tikzcd}
h_C \arrow[r] & X \\
h_D \arrow[r] \arrow[u, "\Phi'"] & Z \arrow[u, "\Phi"']
\end{tikzcd}$$

$$
\begin{tikzcd}
 & p^*(X) \arrow[rr, "\alpha_X"] & & q^*(X) \\
p^*(h_C) \arrow[ur] \arrow[rr] & & q^*(h_C) \arrow[ur] & \\
 & p^*(Z) \arrow[uu, dashed] \arrow[rr, dashed, "\alpha_Z"] & & q^*(Z) \arrow[uu, "q^*(\Phi)"'] \\
p^*(h_D) \arrow[uu] \arrow[ur, dashed] \arrow[rr] & & q^*(h_D) \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$

In the last diagram the top and bottom squares are commutative by definition of F, whereas the front square is commutative by hypothesis. The lateral squares commute because p^* and q^* are functors. The result follows by chasing ζ around the diagram. It is easy to check that $F \circ Y = f$.

To show that Y^* is essentially surjective, we assume that $M \in Filt(\mathbb{C})$ is given and define $\overline{M}(\Phi) = colim(Fig^M \xrightarrow{U^M} \mathbb{C}^{op} \xrightarrow{\Phi} Sets)$. Since Fig^M is filtered, $colim : Sets^{Fig^M} \longrightarrow Sets$ is exact. In particular

$$colim(\Phi \circ U^M \times \Psi \circ U^M) = colim\Phi \circ U^M \times colim\Psi \circ U^M$$

The following computation show that $\Phi \circ U^M \times \Psi \circ U^M = \Phi \times \Psi \circ U^M$:

$$
\begin{aligned}
(\Phi \circ U^M \times \Psi \circ U^M)(h^A \xrightarrow{a} M) &= \Phi \circ U^M(h^A \xrightarrow{a} M) \times \Psi \circ U^M(h^A \xrightarrow{a} M) \\
&= \Phi(A) \times \Psi(A) \\
&= (\Phi \times \Psi)(A) \\
&= (\Phi \times \Psi \circ U^M)(h^A \xrightarrow{a} M)
\end{aligned}
$$

Hence,

$$colim(\Phi \times \Psi \circ U^M) = colim\Phi \circ U^M \times colim\Psi \circ U^M$$

or, what amounts to the same, $\overline{M}(\Phi \times \Psi) = \overline{M}(\Phi) \times \overline{M}(\Psi)$. In the same way we prove that $\overline{M}$ preserves $\mathbb{1}$ and pullbacks. So we conclude that $\overline{M}$ is exact.

Let us prove that $\overline{M}$ has a right adjoint R (thus $(\overline{M}, R)$ is a point).

$$\frac{M \xrightarrow{\alpha} N}{colim M(C) \overset{\overline{\alpha}}{\to} colim N(C)} \quad \downarrow$$

where $C \in Fig_\Phi$. If we have

$$\begin{tikzcd} C \arrow[d] \\ C' \end{tikzcd} \qquad \begin{tikzcd} M(C) \arrow[r, "\alpha_C"] \arrow[d] & N(C) \arrow[d] \arrow[dr, "\eta_C"] & \\ & & colim N(C) \\ M(C') \arrow[r, "\alpha_{C'}"'] & N(C') \arrow[ur, "\eta_{C'}"'] & \end{tikzcd}$$

where α_C and $\alpha_{C'}$ are natural transformations then

$$\begin{tikzcd} M(C) \arrow[dd] \arrow[dr] \arrow[drr] & & \\ & colim M(C) \arrow[r, dashed, "\overline{\alpha}"] & colim N(C) \\ M(C') \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

To finish the proof, we check that $\overline{M} \circ Y = M$ by computing

$$\begin{aligned} \overline{M}(h_C) &= colim_{h^{C'} \longrightarrow M} h_C(C') \\ &= colim_{h^{C'} \longrightarrow M} \mathbb{C}(C', C) \\ &= M(C) \end{aligned}$$

The last equality may be justified by checking that the family

$$(\mathbb{C}(C', C) \xrightarrow{ev_{a'}} M(C))_{(h^{C'} \xrightarrow{a'} M)}$$

is a colimit, a task that is left to the reader.

Alternatively, we could use the commutation lemma 13.1.5

$$\begin{array}{ccccc} Fig^{h_C} & \xrightarrow{U_{h_C}} & \mathbb{C} & \xrightarrow{M} & Sets \\ (C' \to C) & \longmapsto & C' & \longmapsto & M(C') \end{array}$$

$M(C) = colim_{C' \to C} M(C') = \overline{M}(h_C)$. Hence $\overline{M}(h_C) = M(C)$.

We remark that in all the examples with $\mathbb{C}$ being finite, the category of filtering functors is equivalent to the dual of $\mathbb{C}$. Is this always so? Not quite but almost, as we shall see in theorem 14.1.3.

The following, however holds in general

Proposition 14.0.6 *For every small category $\mathbb{C}$, the category $Filt(\mathbb{C}, Sets)$ of filtering functors contains the representables and is closed under filtered colimits and retractions.*

Proof.
To show that h^C is a filtering functor for all $C \in Ob(\mathbb{C})$, we verify the conditions (i)-(iii) of the definition:
(i) $1_C \in h^C(C)$, hence it is not empty.
(ii) Given two figures $h^A \xrightarrow{i} h^C$ and $h^B \xrightarrow{j} h^C$ of h_C we have to find another one, let us say k such that $k \circ i = k \circ j$. We see that by taking $k = 1_C$ we obtain the desired commutativity:

$$\begin{tikzcd} h^A \arrow[r, "i"] \arrow[d, "i"] & h^C \\ h^C \arrow[r, "j"'] \arrow[ur, "k=1_C"] & h^B \arrow[u, "j"'] \end{tikzcd}$$

(iii) Given two figures $h^A \xrightarrow{i} h^C$ and $h^B \xrightarrow{j} h^C$ of h^C we have to find another one, let us say k $\quad i \overset{u}{\underset{v}{\rightrightarrows}} j \overset{w}{\dashrightarrow} k \quad$ such that $w \circ u = w \circ v$ We take $w = j$ and $k = 1_C$ and we obtained the desired commutativity

$$\begin{tikzcd} h^A \arrow[r, "i"] \arrow[dr, shift left, "u"] \arrow[dr, shift right, "v"'] & h^C & \\ & h^B \arrow[u, "j"] \arrow[r, "j"'] & h^C \arrow[ul, "1_C"'] \end{tikzcd}$$ $\square$

We now check the clause on closure under filtered colimits. Let $\mathbb{I}$ be a small filtered category. Consider the following diagram

$$\mathbb{I} \xrightarrow{N} Filt(\mathbb{C}, Sets) \hookrightarrow Sets^{\mathbb{C}}$$

and let $M = colimN \in Sets^{\mathbb{C}}$. We show that M belongs to $Filt(\mathbb{C}, Sets)$ by checking the conditions (i) and (ii) of the second group. The proofs of the other conditions are left to the reader.
(i) We have to show that for a given A, $M(A) \neq \emptyset$. Since $\mathbb{I}$ is filtered, $\exists i_0 \in \mathbb{I}$ such that $N(i_0)$ is a filtering functor. Hence $\exists A$ such that $N(i_0)(A) \neq \emptyset$ and thus $M(A) \neq \emptyset$.
(ii) Let $x \in M(A)$, $y \in M(B)$. Hence $x = [(i, a)]$ and $y = [(j, b)]$ with $a \in N_i(A)$ and $b \in N_j(B)$. Since $\mathbb{I}$ is filtered $\exists k$ $\begin{smallmatrix} i \searrow \\ j \nearrow \end{smallmatrix} k$. So we have that

$$a \in N_i(A) \mapsto a' \in N_k(A) \mapsto x \in M(A)$$

and

$$b \in N_i(B) \mapsto b' \in N_k(B) \mapsto y \in M(B)$$

Since N_k is filtering we conclude that $\exists C$ $C \begin{smallmatrix} \overset{f}{\nearrow} A \\ \underset{g}{\searrow} B \end{smallmatrix}$ and $\exists c' \in N_k(C)$ such that $c' \mapsto a'$ and $c' \mapsto b'$. Take $z = [(k, c')] \in M(C)$.

The clause about closure under retracts is left to the reader. $\Box$

Corollary 14.0.7 *Let $\mathbb{C}$ be a small category. Then the category $Pts(Sets^{\mathbb{C}^{op}})$ of points contains all evaluations, is closed under filtered colimits and retracts*

Proof.
Just notice that under the correspondence between points and filtering functors, h^C corresponds to the evaluation ev_C defined by $ev_C(X) = X(C)$ on objects.

Let us notice that $Fig(h^C)$ has a terminal object, 1_C.

$$\begin{tikzcd} h^A \arrow[r, "i"] \arrow[dr, dashed, "\exists!"'] & h^C \\ & h^C \arrow[u, "1_C"'] \end{tikzcd}$$

This property characterizes representables:

Proposition 14.0.8 *Let* $\mathbb{C} \xrightarrow{M} Sets$*,* M *is representable iff* Fig^M *has a terminal object.*

Proof.
$(\rightarrow)$: has been proved already.
$(\leftarrow)$: let $h^A \xrightarrow{t} M$ a terminal figure. Let us show that t is an isomorphisms. We define $M \xrightarrow{\Phi} h^A$ as follows. Let $\Phi_C(i)$ be the only $A \xrightarrow{f} C$ such that

$$\frac{M(f)(t) = i}{t \circ h^f = i}$$

We have

$$\begin{tikzcd} h^A \arrow[r, "t"] & M \\ h^C \arrow[u, dashed, "\exists!"] \arrow[ur, "i"'] & \end{tikzcd}$$

We have to show that $t \circ \Phi = \Phi \circ t = Id$

$$\frac{h^A \xrightarrow{t} M \xrightarrow{\Phi} h^A}{t \in M(A) \xrightarrow{\Phi_A} \mathbb{C}(A, A)}$$

$\Phi_A(t)$ is the unique $A \xrightarrow{f} A$ So $f = 1_A$. Hence $t \circ h^f = t$, $t = 1_A$ and $\Phi_A(t) = 1_A$

$$\frac{M \xrightarrow{\Phi} h^A \xrightarrow{t} M}{M(C) \xrightarrow{\Phi_C} h^A(C) \xrightarrow{t_C} M(C)}$$

$\Phi_C(i)$ is the unique $A \xrightarrow{f} A$ such that

$$\frac{t \circ h^f = i}{M(f)(t) = i}$$

Hence

$$t_C(\Phi_C(i)) = t_C(f) = M(f)(t_A(1_A)) = i$$

`EXERCISE 14.0.3`

(1) Show that $Pts(Sets^{Sets_0}) \simeq Sets$, where $Sets_0$ is the full subcategory of $Sets$ consisting of the finite sets.

(2) Show that $Pts(Sets^{{Sets_0}^{op}}) \simeq Boole$, where $Boole$ is the category of Boolean algebras.

WINDOW 14.0.2

Algebraic Theories

The following is a generalization of the two previous exercises: let T be an equational theory (the theory of groups, of rings, of Boolean algebras, etc.) and let $Algebras(T)$ be the category of algebras of T (category of groups with group homomorphisms, category of rings with ring homomorphisms, category of Boolean algebras with boolean homomorphisms, etc.) Let $\mathbb{T}$ be the category of finitely presented algebras. Then

$$Filt(\mathbb{T}, Sets) \simeq Points(Sets^{\mathbb{T}}) \simeq Algebras(T).$$

Recall that an algebra A is *finitely presentable* if $h^A : Algebras(T) \longrightarrow Sets$ commutes with filtered colimits. Then $\mathbb{T}^{op}$ is finitely complete and

$$Filt(\mathbb{T}^{op}, Sets) \simeq Exact(\mathbb{T}^{op}, Sets).$$

Notice that this approach opens the door to the study of algebras in arbitrary categories with finite limits. This is the beginning of the categorical approach to Universal Algebra pioneered by Lawvere and Bénabou and others. From the vast literature on this subject, we shall only mention [4], [5], [2], [22], [36] and [43].

14.1 Categories and theories

With any small category $\mathbb{C}$ we shall associate a many sorted language $L_{\mathbb{C}}$ and a theory $T_{\mathbb{C}}$ as follows: the sorts are the objects of $\mathbb{C}$ and the functional symbols (also sorted) are the morphisms of $\mathbb{C}$. If $A \xrightarrow{f} B$ is a morphism, then f as a functional symbol has A as sort of its domain and B as sort of its codomain. We shall assume that for every sort A we have an infinite set of variables of that sort.

To define $T_{\mathbb{C}}$ we define first the notions of geometric formula, geometric sequent and geometric theory. We say that a formula is *geometric* if it is built from the atomic formulas (which include $\top$ and $\bot$) by means of $\wedge$, $\bigvee$, $\exists x \in A$, where $x \in A$ is a variable of sort A. A *geometric sequent* is a triple (ϕ, ψ, V), where ϕ and ψ are geometric formulas and V is a finite sequence of variables containing the free variables of ϕ and ψ. The sequence V is called the *context* of the sequent. In the sequel we shall use '$\phi \vdash_V \psi$' instead of '(ϕ, ψ, V)' to denote a geometric sequent. A *geometric theory* is a set of geometric sequents.

Remark 14.1.1 The notion of context is very natural and important. It occurs, for instance, in analytic geometry: in the context (x, y), the equation $x^2 + y^2 = 1$ represents a circle, but in the context (x, y, z) the same equation represents a cylinder.

We take for granted the notion of *satisfaction* and *truth* for formulas and sentences. We say that a sequent (ϕ, ψ, V) is *valid in a model M* if the sentence which results from $(\phi \rightarrow \psi)$ by universally quantifying the variables of V is true in M.

The geometric theory $T_{\mathbb{C}}$ consists of two groups of axioms:

Group I

(i) $\vdash_x 1_A(x) = x$, whenever $A \in \mathbb{C}$ and x is a variable of sort A

(ii) $\vdash_x h(x) = g(f(x))$, whenever $A \xrightarrow{f} B \xrightarrow{g} C$, $h = g \circ f$ and x is a variable of sort A

Group II

(i) $\vdash \bigvee_A \exists x \top$, whenever $A \in \mathbb{C}$ and x is a variable of sort A

(ii) $\vdash_{x,y} \bigvee_{\substack{C \xrightarrow{f} A \\ C \xrightarrow{g} B}} \exists z (f(z) = x \wedge g(z) = y)$ whenever $A,\ B \in \mathbb{C}$, x is a variable of sort A, y a variable of sort B and z a variable of sort C

(iii) $u(x) = v(x) \vdash \bigvee_{\{C \xrightarrow{w} A \overset{u}{\underset{v}{\rightrightarrows}} B\}} \exists z\ (w(z) = x)$ for each $A \overset{u}{\underset{v}{\rightrightarrows}} B$, whenever x is a variable of sort A, y a variable of sort B and z a variable of sort C

We assume known the notion of a model of a many-sorted theory. Alternatively, the reader may take the following proposition as a definition of this notion.

Proposition 14.1.2 $Models(T_{\mathbb{C}}) \simeq Filt(\mathbb{C}, Sets)$

Proof.
Straightforward. Notice that an $L_{\mathbb{C}}$-structure M satisfies the axioms of the group I iff M is a functor. Furthermore, M satisfies the axioms of both groups iff M is a filtering functor. Finally, it is obvious that a morphism between models is precisely a natural transformation.

Before going further, let us calculate the models for the theory associated with the category of generic figures of (1): *Bouquets*, (2): *Graphs*, (3): *Rgraphs* and (4): *Esets*.

To make the formulas more readable, we shall often write '$\alpha \in L$' rather than 'α is a variable of sort L'.

(1) Let x, y be variables of sort V, α, β variables of sort L.

Axioms (Group II)

(i) $\vdash \exists x \top \vee \exists \alpha \top$

(ii) This axiom says that for each pair of object of the category something happens. In this case we must then consider the following pairs: (V, V), (V, L), (L, V) and (L, L) and analyze what the axiom says for each.

Let us consider (V, V):
We have
$$\begin{tikzcd} & V \arrow[dl, "1_V"'] \arrow[dr, "1_V"] & \\ V & & V \end{tikzcd}$$
and $x \in V$, $y \in V$. Then (ii) says that
$\vdash_{x,y} \exists z\,(1_V z = x \wedge 1_V z = y)$

Let us consider (V, L):
We have
$$\begin{tikzcd} & V \arrow[dl, "1_V"'] \arrow[dr, "v"] & \\ V & & L \end{tikzcd}$$
and $x \in V$, $\alpha \in L$. Then (ii) says that
$\vdash_{x,\alpha} \exists z\,(1_V z = x \wedge vz = \alpha)$. For the pair (L, V) we have the same situation.

Let us consider (L, L):
There are two cases since V and L can both satisfy the conditions. So we have
$$\begin{tikzcd} & V \arrow[dl, "v"'] \arrow[dr, "v"] & \\ L & & L \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & L \arrow[dl, "1_L"'] \arrow[dr, "1_L"] & \\ L & & L \end{tikzcd}$$
$\alpha \in L$, $\beta \in L$. Then (ii) says that $\vdash_{\alpha,\beta} \exists z\,(vz = \alpha \wedge vz = \beta) \vee \exists \gamma\,(1_L\gamma = \alpha \wedge 1_L\gamma = \beta)$.

(iii) There are no non trivial parallel arrows, the only parallel ones are given by the identities and are logically valid sequents:

$$\begin{tikzcd} V \arrow[r, "1_V"] & V \arrow[r, shift left, "1_V"] \arrow[r, shift right, "1_V"'] & V \end{tikzcd}$$

$1_V(x) = 1_V(x) \vdash_x \exists y(1_V(y) = x)$ and

$$\begin{tikzcd} V \arrow[r, "v"] & L \arrow[r, shift left, "1_L"] \arrow[r, shift right, "1_L"'] & L \end{tikzcd}, \quad \begin{tikzcd} L \arrow[r, "1_L"] & L \arrow[r, shift left, "1_L"] \arrow[r, shift right, "1_L"'] & L \end{tikzcd}$$

$1_L(\alpha) = 1_L(\alpha) \vdash_\alpha \exists \gamma(1_L(\gamma) = \alpha \wedge 1_L(\gamma) = \alpha) \vee \exists x(v(x) = \alpha \wedge v(x) = \alpha)$

We simplify the formulations of the axioms (of group II):

(i) $\vdash \exists_\alpha \top$. This says that there is an element in L.

(ii) $\begin{cases} \vdash_{x,y} x = y & \text{which says that } V \text{ has at most one element.} \\ \vdash_{x,\alpha} v(x) = \alpha \\ \vdash_{\alpha,\beta} \alpha = \beta. & \text{Thus } L \text{ has to have one element.} \end{cases}$

(iii) These are logically valid sequents.

So $M(V) \xrightarrow{M(v)} M(L) = \mathbb{1}$. There are two models and a morphism between them:

$$\begin{array}{ccc} \emptyset \longrightarrow \mathbb{1} & = & \bullet \\ & & \downarrow \\ \mathbb{1} \longrightarrow \mathbb{1} & = & \bullet \end{array}$$

The models are the $Pts(Bouquets) = \bullet \longrightarrow \bullet$

(2) Let x, y the variables of sort V, α, β the variables of sort A.

Axioms (Group II)

(i) $\vdash \exists \alpha \top$

(ii) $\begin{cases} \vdash_{\alpha,\beta} \alpha = \beta \vee \exists x(s(x) = \alpha \wedge t(x) = \beta) \vee \exists x(s(x) = \beta \wedge t(x) = \alpha) \\ \vdash_{x,\alpha} s(x) = \alpha \vee t(x) = \alpha \\ \vdash_{x,y} x = y \end{cases}$

(iii) $s(x) = t(x) \vdash_x \perp$.

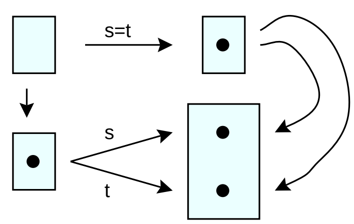
So

$$M(V) \underset{M(t)}{\overset{M(s)}{\rightrightarrows}} M(A)$$

We have then two models and two morphisms between them:

$$\begin{array}{c} \emptyset \xrightarrow{s=t} \mathbb{1} \\ \mathbb{1} \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{2} \end{array}$$

and more graphically



Thus the models of the theory associated to the category of the generic figures of graphs are $Pts(Graphs) = \bullet \rightrightarrows \bullet$

(3) We study now the models of the theory associated with the category of generic figures of the reflexive graphs:

$$\begin{tikzcd} V \arrow[r, shift left=2, "s"] \arrow[r, "t"] & A \arrow[l, shift left=2, "l"] \arrow[loop above, "\sigma"] \arrow[loop below, "\tau"] \end{tikzcd}$$

such that $l \circ s = l \circ t = 1_V$, $\sigma = s \circ l$ and $\tau = t \circ l$. The axioms of group II in this case are:

(i) $\begin{cases} \vdash \exists \alpha \top \\ \vdash \exists x \top \end{cases}$

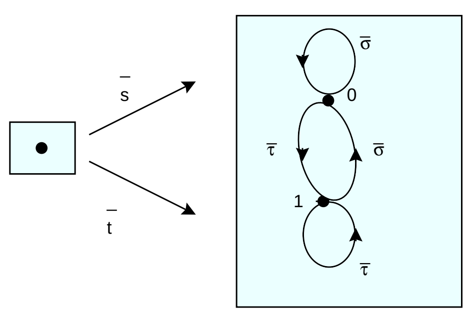
(ii) $\begin{cases} \vdash_{\alpha,\beta}\ \alpha = \beta \vee \sigma(\alpha) = \beta \vee \sigma(\beta) = \alpha \vee \tau(\alpha) = \beta \vee \tau(\beta) = \alpha \\ \vdash_{x,\alpha}\ s(x) = \alpha \vee t(x) = \alpha \vee l(\alpha) = x \vee \exists\gamma(l(\gamma) = x \wedge \sigma(\gamma) = \alpha) \vee \\ \exists\gamma(l(\gamma) = x \wedge \tau(\gamma) = \alpha) \\ \vdash_{x,y}\ x = y \qquad \star \end{cases}$

(iii) There are many parallel arrows $\begin{cases} \sigma(\alpha) = \tau(\alpha) \vdash_\alpha \bot \\ s(x) = t(x) \vdash_x \bot \qquad \star\star \end{cases}$

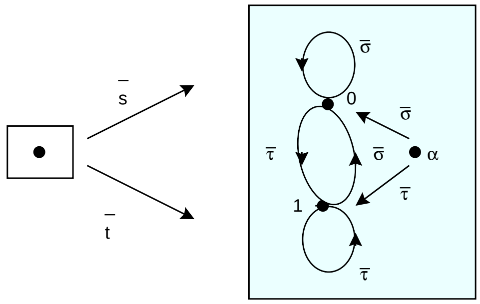
We have then,

$$\begin{tikzcd} M(V) \arrow[r, shift left=2, "\overline{s}"] \arrow[r, "\overline{t}"'] & M(A) \arrow[l, shift left=2, "\overline{l}"] \arrow[loop above, "\overline{\sigma}"] \arrow[loop below, "\overline{\tau}"] \end{tikzcd}$$

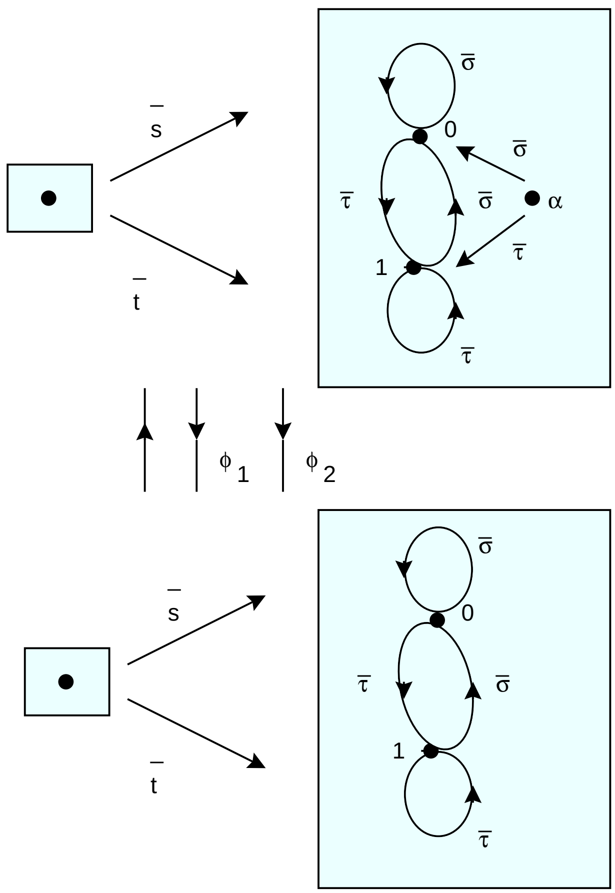
such that $\overline{\sigma}(\overline{s}) = \overline{s}$, $\overline{\sigma}(\overline{t}) = \overline{s}$ and $\overline{\tau}(\overline{s}) = \overline{t}$, $\overline{\tau}(\overline{t}) = \overline{t}$ where for $M(s)$ we write $\overline{s}$, etc. From $\star$ we conclude that $M(V) = \mathbb{1}$ and from $\star\star$ that there are least two elements 0 and 1 in $M(A)$. $\overline{\sigma}$ and $\overline{\tau}$ are two functions defined on 0 and 1: $\overline{\sigma}(0) = 0$, $\overline{\sigma}(1) = 0$, $\overline{\tau}(0) = 1$ and $\overline{\tau}(1) = 1$. Let us represent that with the help of a picture



But there is also another possibility



Hence there are two models with three morphisms between them:



where $\phi_1(\alpha) = 0,\ \phi_1(0) = 0,\ \phi_1(1) = 1, \phi_2(\alpha) = 1,\ \phi_2(0) = 0,$ and $\phi_2(1) = 1$, etc. It is left to the reader to verify that the morphisms between the models

are natural transformations. The models of the theory of *Rgraphs* are the $Pts(Rgraphs) = \bullet \substack{\longrightarrow \\ \longrightarrow \\ \longleftarrow} \bullet$

(4) The axioms of group II of the theory $T_{\mathbb{E}}$ are (with x and y variables of sort $*$) are:

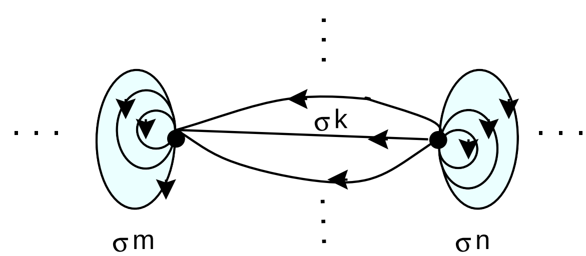
(i) $\vdash \exists x \top$

(ii) $\vdash_{x,y} \bigvee_n \sigma^n(x) = y \vee \sigma^n(y) = x$

(iii) $\sigma^n(x) = \sigma^m(x) \vdash_x \perp$ when $n \neq m$.

The theory $T_{\mathbb{E}}$ has two models:
• $(\mathbb{Z}, \sigma) \xrightarrow{\sigma^m} (\mathbb{Z}, \sigma)$ $m \in \mathbb{Z}$ $\ldots -1 \to 0 \to 1 \to 2 \to \ldots$.
• $(\mathbb{N}, \sigma) \xrightarrow{\sigma^n} (\mathbb{N}, \sigma)$ $n \in \mathbb{N}$ $0 \to 1 \to 2 \to 3 \ldots$
with an infinity of morphisms between these two models.

Thus the models of $T_{\mathbb{E}}$ (or, what amounts to the same, the points of the evolutive sets) are:



where $n \in \mathbb{N}$, $m \in \mathbb{Z}$ and $k \in \mathbb{Z}$.

The following is the promised example of a finite category $\mathbb{C}$ whose filtering functors are not all representable.

Let $\mathbb{C}$ be the category:

$$A \xrightarrow{e} A \substack{\xrightarrow{f} \\ \xrightarrow{g} \\ \xrightarrow[k]{}} B$$

such that $e^2 = e$, $fe = ge = ke = k$.

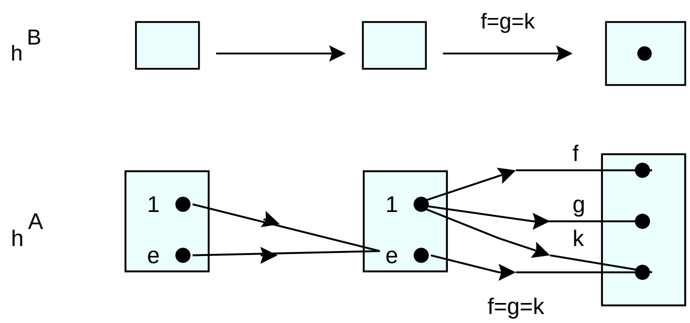
Let us describe the Group II axioms of $T_{\mathbb{C}}$. (We assume that x, y, z are variables of sort A whereas α, β are variables of sort B).

(i) $\vdash \exists \alpha \top$

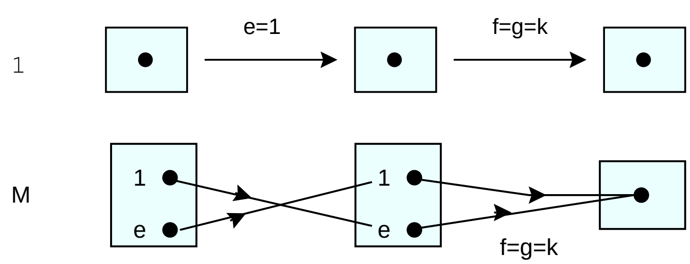
$$\text{(ii)} \begin{cases} \vdash_{x,y} x = y \vee e(x) = y \vee x = e(y) \\ \vdash_{\alpha,\beta} \alpha = \beta \quad \vee \quad \exists x (f(x) = \alpha \wedge g(x) = \beta) \quad \vee \ldots. \\ \qquad\qquad\quad \vee \quad \exists x (f(x) = \alpha \wedge k(x) = \beta) \quad \vee \ldots. \\ \qquad\qquad\quad \vee \quad \exists x (g(x) = \alpha \wedge k(x) = \beta) \quad \vee \ldots. \end{cases}$$

$$\text{(iii)} \begin{cases} f(x) = g(x) \vdash_x \exists z\ e(z) = x \\ g(x) = k(x) \vdash_x \exists z\ e(z) = x \\ f(x) = k(x) \vdash_x \exists z\ e(z) = x \end{cases}$$

There are two representable functors which give two models h^B and h^A with three morphisms $h^f, h^g, h^k : h^B \longrightarrow h^A$ and one morphism $h^A \xrightarrow{h^e} h^A$



Furthermore there are two non representable models: $\mathbb{1}$ (a retract of h^A) and M, described below



Notice, however, that M does not satisfy the axiom $e^2 = e$ of Group I. In fact, $e^2 = 1$ in M.

Summarizing, the category of models and morphisms may be described by the following diagram

$$\begin{tikzcd}
h^B \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[dr] & h^A \arrow[r] \arrow[d, shift right] & h^A \\
& \mathbb{1} \arrow[u, shift right] &
\end{tikzcd}$$

Notice that this category, although not equivalent to the original one, is equivalent to the dual of the Cauchy completion of the original. This is not an accident. In fact, we have the following

Theorem 14.1.3 *Let $\mathbb{C}$ be a finite category (i.e., whose set of morphisms is finite). Then the category of points of $Sets^{\mathbb{C}^{op}}$ is equivalent to the finite category $\overline{\mathbb{C}}^{op}$, the dual of the Cauchy completion of $\mathbb{C}$.*

Proof.
See exercise in [1, p.417]. Although we shall not give the proof, we shall reformulate it so as to understand its meaning and help the reader who would like to find a proof.

Notice that it follows from proposition 5.2.1 and theorem 14.0.5 that

$$Pts(Sets^{\mathbb{C}^{op}}) \simeq Pts(Sets^{\overline{\mathbb{C}}^{op}}) \simeq Filt(\overline{\mathbb{C}}, Sets)$$

Thus, the theorem is equivalent to the statement that $\overline{\mathbb{C}}^{op} \simeq Filt(\overline{\mathbb{C}}, Sets)$. By noticing that the Cauchy completion of a finite category is finite, one may assume (by changing the notation) that $\mathbb{C}$ is Cauchy complete.

Let $Y : \mathbb{C}^{op} \longrightarrow Sets^{\mathbb{C}}$ be the Yoneda functor. Since its values are filtering functors (see section 14), Y may be considered as a functor

$$Y : \mathbb{C}^{op} \longrightarrow Filt(\mathbb{C}, Sets)$$

The theorem says that Y is an equivalence of categories or, what is the same, that Y is essentially surjective, since Y is full and faithful.

Corollary 14.1.4 *Let $\mathbb{C}$ be a finite category. Then if p is a point of $Sets^{\mathbb{C}^{op}}$, the set $p^*(C)$ is finite for every $C \in \mathbb{C}$.*

Proof.
Since $Pts(Sets^{\mathbb{C}^{op}}) \simeq Filt(\mathbb{C}, Sets)$, it is enough to show that for every

filtering functor M and every object A of $\mathbb{C}$, the set $M(A)$ is finite. This is obvious, since every filtering functor is representable by some $C \in \overline{\mathbb{C}}$, which is again a finite category.

We shall give an alternative proof (suggested by R. Lewin) that uses only property (ii) of the definition of a filtering functor.

Let $(a_0, A_0), (a_1, A_1), \ldots$ be an enumeration of a countable subset of the set $|Fig(M)|$. By using property (ii) of the definition of a filtering functor repeatedly, we construct the diagram

$$\begin{tikzcd}
(a_0, A_0) \arrow[dr] & & & \\
(a_1, A_1) \arrow[r] & (b_0, B_0) \arrow[dr] & & \\
(a_2, A_2) \arrow[rr] & & (b_1, B_1) \arrow[dr] & \\
\vdots & & & \ddots
\end{tikzcd}$$

Since the set of objects of $\mathbb{C}$ is finite, some object C appears infinitely often in the diagonal. Furthermore, since the set of morphisms is finite, at least one of the compositions of intermediate morphisms between consecutive C's, say f, appears infinitely often. Furthermore, $f^n = f^m$ for some $n < m$. Thus, we have a diagram

$$(c_0, C) \xrightarrow{f} (c_1, C) \xrightarrow{f} (c_2, C) \ldots$$

i.e., such that $M(f)(c_1) = c_0, M(f)(c_2) = c_1 \ldots$. We claim that the set $\{c_0, c_1, c_2 \ldots\}$ is finite. This is obvious, since 'below' any c_i there are at most $n+1$ elements. This clearly implies that the enumeration of $|Fig(M)|$ is finite. $\square$

The interest of the previous corollary lies in the following consequence: we cannot describe theories having at least one infinite model in terms of finite categories. One possibility exploited in the theory of sketches is to describe these theories in terms of finite graphs with finitely many stipulations and finitely many 'axioms'. (See [33])

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