General Relativity: The Riemann-Christoffel tensor

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0.1 Curvature of a connection

Following Lavendhomme [1] we define the curvature of the connection ∇ as the map $R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ defined by

$$R_{XY}Z = (C \circ C^D - C \circ C^D \circ \Sigma)(X \star Y \star Z)$$

where $X, Y, Z \in \mathcal{X}(M)$, $C : M^{D \times D} \longrightarrow M^D$ is the connection map of ∇ and $(X \star Y \star Z)(d_1, d_2, d_3) = Z_{d_3} \circ Y_{d_2} \circ X_{d_1}$

In the following proposition, $\sum_0$ denotes the "cyclic sum" defined by $\sum_0 \phi(X_1, X_2, X_3) = \phi(X_1, X_2, X_3) + \phi(X_2, X_3, X_1) + \phi(X_3, X_1, X_2)$

Proposition 0.1

1. *$R_{XY}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$*
2. *$R_{X_2 X_1} X_3 = -R_{X_1 X_2} X_3$*
3. *Assume that ∇ is symmetric. Then $\sum_0 R_{X_1 X_2} X_3 = 0$. (First Bianchi identity).*

Proof:

1. Already done (cf. "Curvaturetorsion.pdf" Proposition 0.14, page 12)

2. Immediate from the first and $\nabla_{-X}Y = -\nabla_X Y$. (cf. "Covariantderivation.pdf" Theorem 3, page 3)

3. Since ∇ is symmetric, $\nabla_X Y - \nabla_Y X = [X, Y]$ (cf. "Curvaturetorsion.pdf" Proposition 0.15, page 16). Compute the LHS using this relation repeatedly

$$\begin{aligned}
\textstyle\sum_0 R_{X_1 X_2} X_3 &= \textstyle\sum_0 (\nabla_{X_1}\nabla_{X_2} X_3 - \nabla_{X_2}\nabla_{X_1} X_3 - \nabla_{[X_1,X_2]} X_3) \\
&= \textstyle\sum_0 \nabla_{X_1}[X_2, X_3] - \sum_0 \nabla_{[X_1,X_2]} X_3 \\
&= \textstyle\sum_0 (\nabla_{X_1}[X_2, X_3] - \nabla_{[X_2,X_3]} X_1) \\
&= \textstyle\sum_0 [X_1, [X_2, X_3]] \\
&= 0 \text{ (Jacobi's identity)}
\end{aligned}$$

0.2 Curvature of a connection when $M = R^n$.

We start with the relation

$$R_{\partial/\partial x^i \partial/\partial x^j} \partial/\partial x^k = \nabla_{\partial/\partial x^i}(\nabla_{\partial/\partial x^j} \partial/\partial x^k) - \nabla_{\partial/\partial x^j}(\nabla_{\partial/\partial x^i} \partial/\partial x^k)$$

which follows from 0.1 and $[\partial/\partial x^i, \partial/\partial x^j] = 0$.

We now use the relation

$$\nabla_{\partial/\partial x^i} \partial/\partial x^j = \sum_k \Gamma^k_{ij} \partial/\partial x^k$$

(cf, "Covariantderivation.pdf" page 4). Therefore,

$$\begin{aligned}
(1) \quad \nabla_{\partial/\partial x^i}(\textstyle\sum_l \Gamma^l_{jk} \partial/\partial x^l) &= \textstyle\sum_l \nabla_{\partial/\partial x^i}(\Gamma^l_{jk} \partial/\partial x^l) \\
&= \textstyle\sum_l (\partial\Gamma^l_{jk}/\partial x^i \partial/\partial x^l + \Gamma^l_{jk} \nabla_{\partial/\partial x^i} \partial/\partial x^l) \\
&= \textstyle\sum_l \partial\Gamma^l_{jk}/\partial x^i \partial/\partial x^l + \sum_l \Gamma^l_{jk} \nabla_{\partial/\partial x^i} \partial/\partial x^l
\end{aligned}$$

Similarly,

$$(2) \quad \nabla_{\partial/\partial x^j}(\textstyle\sum_l \Gamma^l_{ik} \partial/\partial x^l) = \sum_l \partial\Gamma^l_{ik}/\partial x^j \partial/\partial x^l + \sum_l \Gamma^l_{ik} \nabla_{\partial/\partial x^j} \partial/\partial x^l$$

Letting $R^k_{ij} = R_{\partial/\partial x^i \partial/\partial x^j} \partial/\partial x^k$ we have

$$R^k_{ij} = (1) - (2)$$

F and its l^{th} component is readily computed to be

$$(R^k_{ij})_l = \partial\Gamma^l_{jk}/\partial x^i - \partial\Gamma^l_{ik}/\partial x^j + \sum_\alpha(\Gamma^\alpha_{jk}\Gamma^l_{i\alpha} - \Gamma^\alpha_{ik}\Gamma^l_{j\alpha})$$

This coincides with what Spivak (vol II, page 189) and Einstein ("The meaning of Relativity", page 74, fifth edition) call R^l_{kij}, but withΓ^l_{ji} instead of our Γ^l_{ij}. Notice, however, that this difference is spurious. Since ∇ is symmetric, $\Gamma^k_{ij} = \Gamma^k_{ji}$ as the following argument shows:

$$\nabla(t_1, t_2)(d_1, d_2) = x + d_1e_i + d_2e_j + d_1d_2\nabla_4((x, e_i), (x, e_j))$$

Therefore,

$$\nabla(t_2, t_1)(d_2, d_1) = x + d_2e_j + d_1e_1 + d_1d_2\nabla_4((x, e_j), (x, e_i))$$

By definition of ∇ being symmetric, $\nabla(t_1, t_2)(d_1, d_2) = \nabla(t_2, t_1)(d_2, d_1)$ and this implies that $\Gamma^k_{ij} = -(\nabla_4(x, e_j), (x, e_i)))_k = -(\nabla_4(x, e_i), (x, e_j)))_k = \Gamma^k_{ji}$.

0.3 The Riemann-Christoffel tensor

Let M be microlinear with a symmetric connection ∇. In this section we define the tensor of the title in terms of the curvature R of ∇.

We define the *Riemann-Christoffel tensor* to be the map $R : M^D \times_M \times M^D \times_M M^D \longrightarrow M^D$ given by the prescription

$$R(t_1, t_2, t_3) = (R_{X_1X_2}X_3)_m$$

where $t_1, t_2, t_3 \in M_m$ and $(X_i)_m = t_i$.

For this definition to make sense, we must show the following

Lemma 0.2 *Assume that $X_m = X'_m, Y_m = Y'_m, Z_m = Z'_m$. Then*

$$(R_{XY}Z)_m = (R_{X'Y'}Z')_m$$

Proof: Notice first that $(R_{XY}Z)_m = (R_{X'Y}Z)_m$, i.e., a letter may be replaced by the corresponding primed letter if it occurs in first position. This follows from the definition of the curvature, namely

$$R_{XY}Z = (C \circ C^D - C \circ C^D \circ \Sigma)(Z \star Y \star X)_m$$

by observing that $(Z \star Y \star X)_m = (Z \star Y \star X')_m$. In fact,

$$\begin{aligned}
(Z \star Y \star X)_m(d_1, d_2, d_3) &= (Z_{d_3} \circ Y_{d_2} \circ X_{d_1})_m \\
&= (Z_{d_3} \circ Y_{d_2})(X_m(d_1)) \\
&= (Z_{d_3} \circ Y_{d_2})(X'_m(d_1)) \\
&= (Z_{d_3} \circ Y_{d_2} \circ X'_{d_1})_m \\
&= (Z \star Y \star X')_m(d_1, d_2, d_3)
\end{aligned}$$

Using the previous proposition,

$$\begin{aligned}
R_{X'Y'}Z' &= -R_{Y'Z'}X' - R_{Z'X'}Y' \\
&= R_{Z'Y'}X' - R_{Z'X'}Y' \\
&= R_{ZY'}X' - R_{ZX'}Y' \\
&= -R_{Y'Z}X' - R_{ZX'}Y' \\
&= R_{X'Y'}Z \\
&= R_{XY}Z
\end{aligned}$$

Proposition 0.3 *The Riemann-Christoffel tensor has the following properties*

$$\begin{aligned}
&R(t_2, t_1, t_3) = -R(t_2, t_1, t_3) \\
&R(t_1, t_2, t_3) + R(t_2, t_3, t_1) + R(t_3, t_1, t_2) = 0 \\
&R_(\lambda t_1, t_2, t_3) = \lambda R(t_1, t_2, t_3) \\
&R_(t_1, \lambda t_2, t_3) = \lambda R_(t_1, t_2, t_3) \\
&R_(t_1, t_2, \lambda t_3) = \lambda R_(t_1, t_2, t_3)
\end{aligned}$$

Proof: Immediate from the corresponding ones for $R_{XY}Z$ in "curvaturetorsion.pdf" page 17.

Conjecture: $R(t_1, t_2, t_3) = R_{t_1 t_2} t_3$, where the RHS refers to the Riemann-Christoffel tensor defined à la Cartan (cf. mo/re and ???)

For finite dimensional manifolds, this follows from the Proposition in page 236 of Moerdijk/Reyes. However, no proof of this proposition is given beyond the comment that it follows from "a horrible computation".

0.4 Contraction of a tensor

PUT THIS IN CLASSICAL CONTEXT ONLY! If R^l_{ijk} is the curvature tensor, we define $R_{ij} = \sum_l R^l_{ijl}$ to be the Ricci tensor. The operation of the passage from the first tensor to the second is called "contraction".

From a more conceptual point of view, the Riemann-Christoffel curvature tensor is given by a map $R : M^D \times_M M^D \times_M M^D \longrightarrow M^D$. The contraction is a new map $Ric : M^D \times_M M^D \longrightarrow M^D$ defined as follows: if $u, v \in M_x$, let $\Psi(u,v) : M_x \longrightarrow M_x$ be the R-linear map defined by $\Psi(u,v)(w) = R_{uw}v$. Then $Ric(u,v) = trace(\Psi(u,v))$.

To connect this invariant definition of Ric with the classical expression of the Ricci tensor we take any basis of M_x to represent $\Psi(u,v)$ as a matrix. Then $Ric(u,v)$ is the trace of this matrix. Take, for instance the basis $\{\partial/\partial x^i|_x\}_i$. Then the matrix of $\Psi(u,v)$ relative to this basis is $(\omega^i(\Psi(u,v)(\partial/\partial x^j|_x))_{ij}$, where $\{\omega^i\}_i$ is the dual basis. Then its trace is $Ric(u,v) = \sum_i \omega^i(R_{u(\partial/\partial x^i|_x)}v)$. In coordinates, if $u = \partial/\partial x^j|_x$ and $v = \partial/\partial x^i|_x$, then the Riemann-Christoffel tensor is R^l_{ijk} and the Ricci tensor is indeed $\sum_l R^l_{ijl} = R_{ij}$, i.e., its contraction.

Proposition 0.4 *If ∇ is symmetric, then*

$$Ric(u,v) = Ric(v,u)$$

Proof: From proposition ?? $R_(u, \partial/\partial x_i|_x) + R(\partial/\partial x_i|_x, v, u) + R(v, u, \partial/\partial x_i|_x) = 0$. Thus $R(u, \partial/\partial x_i|_x, v) - R(v, \partial/\partial x_i|_x, u) = R(u, v, \partial/\partial x_i|_x)$ which in turn implies that

$$Ric(u,v) - Ric(v,u) = \sum_i \omega^i R(u, v, \partial/\partial x_i|_x)$$

We are reduced to prove

Claim: $\sum_i \omega^i R(u, v, \partial/\partial x_i|_x) = 0$

Another way of posing the problem: let $\phi : M_x \longrightarrow M_x$ be the linear transformation defined by $\phi(w) = R(u,v,w)$. Then its matrix is skew symmetric. In fact ???

REST OF THE PROOF???

I cannot prove this proposition, but something much weaker:

Lemma 0.5 *For every $x \in R$, $x^2 - x + 1 > 0$.*

Proof: LATER!

Lemma 0.6 *Let $v_1, \ldots, v_n \in R$. There exists $v_0 > 0$ such that for every i $v_0 + v_i$ and $v_0 - v_i$ are units.*

Proof: Take $v_0 = 1 + \sum_i v_i^2$ and use previous lemma.

Corollary 0.7 *Let $\phi : R^n \times R^n \longrightarrow R$ be multilinear such that $\phi(u, u) = 0$ whenever $u \neq 0$. Then ϕ vanishes on the diagonal.*

Proof: Let $v \in R^n$. Then $v = (v_1 \ldots, v_n)$. By the previous lemma there is v_0 such that both $v_0 + v_i$ and $v_0 - v_i$ are units. Let $u = (v_0, \ldots, v_0)$. Hence u, $u + v$ and $u - v$ are non-zero. Thus

$$0 = \phi(u + v, u + v) = \phi(u, u) + \phi(u, v) + \phi(v, u) + \phi(v, v)$$
$$0 = \phi(u - v, u - v) = \phi(u, u) + \phi(u, -v) + \phi(-v, u) + \phi(-v, -v)$$

Adding these two equations and using multilinearity we obtain the desired result. CHECK!

Einstein' vacuum field equations

$$Ric(u, u) = 0 \text{ for every } u \in M_x$$

To conclude $Ric = 0$ it is enough to show that Ric is symmetric. I don't know whether this can be proved in this general context. However, by defining the symmetrization of Ric, $Ric^{sym}(u, v) = 1/2(R(u, v) + R(v, u))$ we can write these equations as

$$Ric^{sym} = 0.$$

(In fact, develop $Ric(u + v, u + v) = Ric(u, u) + 2Ric^{sym}(u, v) + Ric(v, v)$)

References

[1] R.Lavendomme, *Basic Concepts of Synthetic Differential Geometry*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1996

[2] I.Moerdijk and G.E.Reyes