Ordinary differential equations and their exponentials

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Vector fields or, equivalently, ordinary differential equations have long been considered, heuristically, to be the same as "infinitesimal (pointed) actions" or "infinitesimal flows", but it is only with the development of Synthetic Differential Geometry (SDG) that we have the tools to formulate these notions and prove their equivalence in a rigourous mathematical way. We exploit this fact to define the exponential of two ordinary differential equations as the exponential of the corresponding infinitesimal actions. The resulting action is seen to be the same as a partial differential equation whose solutions may be obtained by conjugation from the solutions of the differential equations that make up the exponential. Furthermore, we show that this method of conjugation is equivalent, under some conditions, to the method of change of variables, widely used to solve differential equations.

Our paper has two parts: in the first, we study generalities on actions. In the second, we describe the exponential of two such actions to obtain the above mentioned result. Some examples illustrate the general method. A preliminary version of this paper appeared as part of [5].

1 Generalities on actions

Recall that an *action* of a set (object) D on a set (object) M is a map $X : D \times M \to M$, and a *homomorphism* of actions $(M, X) \to (N, Y)$ is a map $f : M \to N$ with $f(X(d, m)) = Y(d, f(m))$ for all $m \in M$ and $d \in D$.

The category of actions by a set D form a topos. We shall be interested in particular in the exponential formation in this topos (cf. section 2).

In the applications below, D is the usual set of square zero elements in R. It is a *pointed* object, pointed by $0 \in D$, and the actions $X : D \times M \to M$ we consider, are *pointed* actions in the sense that $X(0, m) = m$ for all $m \in M$, or equivalently, $X_0 : M \to M$ is the identity map on M. A pointed action, in this situation, is the same thing as a vector field on M, cf. [7].

The pointed actions likewise form a topos, and the exponential to be described in section 2, is the same as the exponential in the category of actions (cf. [4]). If D is the set of square zero elements of R, the exponential of pointed actions may be viewed as the exponential of two vector fields or, alternatively, of two first order ordinary differential equations (1ODE's).

For the case of vector fields seen as actions by D, we want to describe the "streamlines" generated by a vector field in abstract action-theoretic terms; this is going to involve a certain "universal" action $(\tilde{R}, \Delta)$: $\tilde{R}$ is an "infinitesimally open subset" of R, i.e., whenever $x \in \tilde{R}$ then $x + d \in \tilde{R}$ for every $d \in D$. The main examples of such subsets are R itself, the non-negative numbers $R_{\geq 0}$, open intervals, and the set D_∞ of all nilpotent elements of the number line. The action Δ is the vector field $\partial/\partial x : D \times \tilde{R} \to \tilde{R}$ given by $(d, t) \mapsto d + t$. The main property to be assumed is that the individual Δ_d's are homomorphisms of D-actions (which is a commutativity requirement); the structure of $\tilde{R}$ could probably be derived from this, but we shall be content with *assuming* that $\tilde{R}$ is an additively written monoid, and that $D \subseteq \tilde{R}$ (with the 0 of D also being the zero of the monoid).

First, if (M, X) is a set with an action, a homomorphism $f : (\tilde{R}, \Delta) \to (M, X)$ is to be thought of as a particular solution of the differential equation given by X, with initial value $f(0)$, or as a "streamline" for the vector field X, starting in $f(0)$. One wants, however, also to include dependence on initial value into the notion of solution, and so one is led to consider maps

$$F : \tilde{R} \times M \to M,$$

satisfying at least $F(d, m) = X(d, m)$ for all $d \in D$ and $m \in M$; we shall consider and compare the following further conditions (universally quantified over all $d \in D$, $t, s \in \tilde{R}$, $m \in M$):

$$F(\Delta(d,t),m)) = X(d, F(t,m)); \tag{1}$$

this is the main one, the two following conditions are included for systematic reasons only:

$$F(\Delta(d,t),m)) = F(t, X(d,m)), \tag{2}$$

$$F(t, X(d,m)) = X(d, F(t,m)) \tag{3}$$

Finally, one may consider the following equation

$$F(t+s,m) = F(t, F(s,m)). \tag{4}$$

Writing X_d for the map $X(d,-) : M \to M$, and similarly for F, condition (1) may be rewritten as

$$F_{\Delta(d,t)} = X_d \circ F_t$$

The others may be rewritten in a similar way. For instance (4) may be rewritten as

$$F_{t+s} = F_t \circ F_s$$

Equation (1) expresses that, for each fixed $m \in M$, the map $F(-,m) : \tilde{R} \to M$ is a homomorphism (and thus, by virtue of $F(0,m) = m$, a "solution with initial value m"). Writing the action of D in terms of the symbol $\cdot$, we may write it $F(d \cdot t, m) = d \cdot F(t,m)$. Equation (2) expresses a certain bi-homogeneity condition of F, $F(d \cdot t, m) = F(t, d \cdot m)$; (3) says that for fixed $t \in R$, $F(t,-) : M \to M$ is an endomorphism of D-actions, $F(t, d \cdot m) = d \cdot F(t,m)$. Finally (4) is the usual condition for action af a monoid on a set M. Clearly, it implies all the others.

Let X be a vector field on M, thought of as a first-order differential equation and let $\tilde{R}$ be an infinitesimally open subset of R. We say that a map $f : \tilde{R} \longrightarrow M$ is a *particular solution of* $(M, X, \tilde{R})$ if $f_d = X_d$ and f satisfies $f(t+d) = X(d, f(t))$.

We say that a map $F : \tilde{R} \times M \to M$ is a *complete solution of* $(M, X, \tilde{R})$ if $F_d = X_d$ and F satisfies (1).

From now on, we shall usually omit reference to M and $\tilde{R}$, which will be presupposed, and talk of solutions 'of a vector field X'.

A complete solution does not satisfy the other conditions (2)-(4), but it does, provided that X satisfies a certain axiom (reflecting, synthetically, validity of the uniqueness assertion for solutions of differential equations on M). — The axiom in question is the following

Uniqueness property for particular solutions of X:
If X is a D-action on M, and $f, g : \tilde{R} \to M$ are homomorphisms of actions, with $f(0) = g(0)$, then $f = g$.

Note that the validity of the axiom, for a given X, depends on M and the choice of $\tilde{R}, \Delta$. For instance, we shall prove below that it holds for any microlinear M if $\tilde{R}$ is taken to be D_∞ (and $\Delta = \partial/\partial x$).

This axiom has the following simple consequence

Proposition 1.0.1 Uniqueness property for complete solutions of X: *Assume that X has the uniqueness property for particular solutions . Then there is at most one complete solution*

$$F : \tilde{R} \times M \longrightarrow M$$

The converse doesn't seem to be true, but a weaker result is true

Proposition 1.0.2 *Assume that there is a complete solution $F : D_\infty \times M \longrightarrow M$ for a vector field X on M. Then X has the uniqueness property for particular solutions.*

Proof: Let $f : D_\infty \longrightarrow M$ be a solution of X. Then

$$f(t) = F(t, f(0))$$

The proof of this equation proceeds 'by induction': the equation is obviously true for $t = 0$. Assume that it is true for t. We prove that it is true for $t + d$. In fact,

$$f(t + d) = X(d, f(t)) = X(d, F(t, f(0)) = F(t + d, f(0))$$

The first and last equality hold because f and F are solutions, whereas the middle one holds by induction. Uniqueness for X follows at once. Notice the following consequence of this proposition: If there is a complete solution for X, then the solution is unique.

Proposition 1.0.3 *Let X be a vector field on M and assume that X satisfies uniqueness property for particular solutions. Then if $F : \tilde{R} \times M \to M$ is the complete solution of the differential equation X, it satisfies properties (2) and (3). Furthermore, if $\tilde{R}$ is a monoid (under +) then F also satisfies (4).*

Proof. Since the proofs are quite similar, we shall do only (4). Fix $m \in M$ and $s \in \tilde{R}$ and define the couple of functions $f,\ g : \tilde{R} \to M$ by the formulas

$$\begin{cases} f(t) = F_{t+s}(m) \\ g(t) = F_t \circ F_s(m) \end{cases}$$

We have to check that f and g are homomorphisms of D-actions, i.e., they satisfy (1). Let us do this for the first

$$\begin{aligned} f(t+d) &= F_{(t+d)+s}(m) \\ &= F_{d+(t+s)}(m) \\ &= F_d \circ F_{t+s}(m) \\ &= X_d \circ f(t). \end{aligned}$$

The proof that g is a homomorphism is similar. Thus, the equality of the two expressions follows from the uniqueness property for particular solutions assumed for X.

Recall that a vector field X on M is called *integrable* if there exists a complete solution $F : \tilde{R} \times M \to M$. If we assume the uniqueness property for particular solutions, the equation (4) holds; if further the commutative monoid structure + on $\tilde{R}$ actually is a group structure, then (4) implies that the action is invertible, with X_{-d} as X_d^{-1} (in fact $F_{-d} = F_d^{-1}$). Of course, both the uniqueness property and the question whether or not the vector field X is integrable, depends on which $\tilde{R}$ is considered. In particular, we shall say that X is *formally integrable* or *has a formal solution* if X is integrable for $\tilde{R} = D_\infty$ (which is a group under addition). For the case of $M = R^n$, this amounts to integration by formal power series, whence the terminology.

Theorem 1.0.4 *The uniqueness property holds for any vector field on a microlinear object, (for $\tilde{R} = D_\infty$). Furthermore, every vector field on a microlinear object is formally integrable. Thus, every vector field on a microlinear object has a unique formal solution.*

Proof. This theorem was stated in [1] and a sketch of the proof by induction was indicated. We give here a proof in detail which does not use induction.

We need to recall some infinitesimal objects from the literature on SDG, cf. e.g. [6]. Besides $D \subseteq R$, consisting of $d \in R$ with $d^2 = 0$, we have $D^n \subseteq R^n$, the n-fold product of D with itself. It has the subobject $D(n) \subseteq D^n$ consisting of those n-tuples $(d_1, \ldots, d_n)$ where $d_i \cdot d_j = 0$ for all i, j. There is also the object $D_n \subseteq R$ consisting of $\delta \in R$ with $\delta^{n+1} = 0$; D_∞ is the union of all the D_n's. If $(d_1, \ldots, d_n) \in D^n$, then $d_1 + \ldots + d_n \in D_n$.

Now, let M be a microlinear object, and X a vector field on it. We first recall that if $d_1, d_2 \in D$ have the property that $d_1 + d_2 \in D$, then $X_{d_1} \circ X_{d_2} = X_{d_1+d_2}$. (For microlinear objects perceive $D(2)$ to be a pushout over $\{0\}$ of the two inclusions $D \to D(2)$, and clearly both expressions given agree if either $d_1 = 0$ or $d_2 = 0$.) In particular, X_{d_1} and X_{d_2} commute. But more generally,

Lemma 1.0.5 *If X is a vector field on a microlinear object and $d_1, d_2 \in D$, the maps X_{d_1} and X_{d_2} commute.*

Proof. This is a consequence of the theory of Lie brackets, cf. e.g. [6] 3.2.2, namely $[X, X] = 0$.

Likewise

Lemma 1.0.6 *If X is a vector field on a microlinear object and $d_1, \ldots, d_n \in D$ are such that $d_1 + \ldots + d_n = 0$, then*

$$X_{d_1} \circ \ldots \circ X_{d_n} = 1_M$$

(= the identity map on M). In particular, $(X_d)^{-1} = X_{-d}$.

Proof. We first prove that R, and hence any microlinear object, perceives D_n to be the orbit space of D^n under the action of the symmetric group $\mathbf{S}_n$ in n letters: Assume that $p : D^n \to R$ coequalizes the action, i.e. is symmetric in the n arguments. By the basic axiom of SDG, p may be written in the form

$$p(d_1, \ldots, d_n) = \sum_{Q \subseteq \{1,\ldots,n\}} a_Q d^Q$$

for unique a_Q's in R (where d^Q denotes $\prod_{i\in Q} d_i$). We claim that $a_Q = a_{\pi(Q)}$ for every $\pi \in \mathbf{S}_n$. Indeed,

$$\sum_Q a_Q d^Q = \pi(\sum_Q a_Q d^Q)$$

since p is symmetric. But

$$\pi(\sum_Q a_Q d^Q) = \sum a_Q d^{\pi(Q)} = \sum_Q a_{\pi^{-1}(Q)} d^Q.$$

By comparing coefficients and using uniqueness of coefficients, we conclude $a_Q = a_{\pi(Q)}$, and this shows that p is (the restriction to D^n of) a symmetric polynomial $R^n \to R$. By Newton's theorem (which holds internally), p is a polynomial in the elementary symmetric polynomials σ_i. Recall that $\sigma_1(d_1, \ldots, d_n) = d_1 + \ldots + d_n$: and each σ_i, when restricted to D^n, is a function of σ_1, since $d_1^2 = 0$; e.g.

$$\sigma_2(d_1, \ldots, d_2) = \sum d_i d_j = \frac{1}{2}(d_1 + \ldots + d_n)^2 = \frac{1}{2}(\sigma_1(d_1, \ldots, d_n))^2.$$

Now consider, for fixed $m \in M$, the map $p : D^n \to M$ given by $(d_1, \ldots, d_n) \mapsto X_{d_1} \circ \ldots \circ X_{d_n}(m)$. By Lemma 1.0.5, this map is invariant under the symmetric group $\mathbf{S}_n$ (recall that this group is generated by transpositions), so there is a unique $\phi : D_n \to M$ such that

$$\phi(d_1 + \ldots + d_n) = X_{d_1} \circ \ldots \circ X_{d_n}(m).$$

So if $d_1 + \ldots + d_n = 0$, $X_{d_1} \circ \ldots \circ X_{d_n}(m) = \phi(0) = \phi(0 + \ldots + 0) = X_0 \circ \ldots \circ X_0(m) = m$. This proves the Lemma.

We can now prove the Theorem. We need to define $F_t : M \to M$ when $t \in D_\infty$. Assume for instance that $t \in D_n$. By microlinearity of M, M perceives D_n to be the orbit space of D^n under the action of $\mathbf{S}_n$ (see the proof of Lemma 1.0.6), via the map $(d_1, \ldots, d_n) \mapsto d_1 + \ldots + d_n$, so we are forced to define $F_t = X_{d_1} \circ \ldots X_{d_n}$ if F is to extend X and to satisfy (4). The fact that this is well defined independently of the choice of n and the choice of $d_1, \ldots, d_n$ that add up to t follows from Lemma 1.0.6.

As a particular case of special importance, we consider a *linear* vector field on a microlinear and Euclidean R-module V. To say that the vector field is linear is to say that its principal-part formation $V \to V$ is a linear map, Δ, say. We have then the following version of a classical result:

Proposition 1.0.7 *Let a linear vector field on a microlinear Euclidean R-module V be given by the linear map $\Delta : V \to V$. Then the unique formal solution of the corresponding differential equation, i.e., the equation $\dot{F}(t) = \Delta(F(t))$ with initial position v, is the map $D_\infty \times V \to V$ given by*

$$(t, v) \mapsto e^{t\cdot\Delta}(v), \tag{5}$$

where the right hand side here means the sum of the following "series" (which has only finitely many non-vanishing terms, since t is assumed nilpotent):

$$v + t\Delta(v) + \frac{t^2}{2!}\Delta^2(v) + \frac{t^3}{3!}\Delta^3(v) + \dots$$

Here of course $\Delta^2(v)$ means $\Delta(\Delta(v))$, etc.

Proof. We have to prove that $\dot{F}(t) = \Delta(F(t))$. We calculate the left hand side by differentiating the series term by term (there are only finitely many non-zero terms):

$$\Delta(v) + \frac{2t}{2!}\cdot\Delta^2(v) + \frac{3t^2}{3!}\Delta^3(v) + ... = \Delta(v + t\cdot\Delta(v) + \frac{t^2}{2!}\cdot\Delta^2(v) + ...)$$

using linearity of Δ. But this is just Δ applied to $F(t)$.

There is an analogous result for second order differential equations of the form $\ddot{F}(t) = \Delta(F(t))$ (with Δ linear); the proof is similar and we omit it:

Proposition 1.0.8 *The formal solution of this second order differential equation $\ddot{F} = \Delta F$, with initial position v and initial speed w, is given by*

$$F(t) = v + t\cdot w + \frac{t^2}{2!}\Delta(v) + \frac{t^3}{3!}\Delta(w) + \frac{t^4}{4!}\Delta^2(v) + \frac{t^5}{5!}\Delta^2(w) + ...$$

2 Exponential of vector fields

As we mentioned before, we shall describe the exponential of two D-actions. We do this when the action in the exponent is *invertible*. An action X :

$D \times M \to M$ is called *invertible*, if for each $d \in D$, $X(d, -) : M \to M$ is invertible. In this case, the exponential $(N, Y)^{(M,X)}$ may be described as N^M equipped with the following action by D: an element $d \in D$ acts on $\beta : M \to N$ by "conjugation":

$$\beta \mapsto Y_d \circ \beta \circ (X_d)^{-1},$$

where Y_d denotes $Y(d, -) : N \to N$, and similarly for X_d.

It is easy to check that if both actions are pointed, so is the above exponential. This means that exponentials in the category of pointed objects are formed by taking exponentials in the topos of actions.

In this section, we show that solutions of an exponential vector field may be obtained by conjugating solutions of the vector fields that make up the exponential. Furthermore, this method of conjugation is equivalent (under some conditions) to the method of change of variables, widely used to solve differential equations.

Theorem 2.0.9 *Assume that (M, X) and (N, Y) are vector fields having (complete) solutions $F : \tilde{R} \times M \to M$ and $G : \tilde{R} \times N \to N$, respectively, and assume that all F_t are invertible. Then a (complete) solution $H : \tilde{R} \times M \to M$ of the exponential $(N, Y)^{(M,X)}$ is obtained as the map*

$$H : \tilde{R} \times N^M \to N^M$$

given by conjugation: $H_t(\beta) = G_t \circ \beta \circ F_t^{-1}$.

Proof. This is purely formal. For $\beta \in N^M$, we have

$$\begin{aligned} (Y^X)_d(H_t(\beta)) &= Y_d \circ H_t(\beta) \circ X_d^{-1} \\ &= Y_d \circ G_t \circ \beta \circ F_t^{-1} \circ X_d^{-1} \\ &= G_{d+t} \circ \beta \circ F_{d+t}^{-1} \\ &= H_{d+t}(\beta), \end{aligned}$$

where in the third step we used the equation (1) for G and F, in the form

$$G_{d+t} = Y_d \circ G_t, \quad \text{respectively } F_{d+t} = X_d \circ F_t,$$

together with invertibility of F_s for all s and invertibility of X_d.

A similar argument gives that if each of (2)-(4) holds for both F and G, then the corresponding property holds for H.

In most applications, the invertibility of the F_t will be secured by subtraction on $\tilde{R}$, with $F_t^{-1} = F_{-t}$.

Recall that an R-module V is called *Euclidean* if the canonical map $\alpha : V \times V \to V^D$ given by $\alpha(u,v)(d) = u + d \cdot v$ is invertible; the composite of α^{-1} with projection to the second factor, $V^D \to V \times V \to V$ is called *principal part* formation.

If $M \subseteq V$ is an infinitesimal open (definition???) subset of V, the canonical map $M \times V \longrightarrow M^D$ is an isomorphism and a vector field X on M may be seen as a map $X : M \longrightarrow M \times V$. By composing it with the principal part formation for V, we obtain a (non necessarily linear) map $\xi : M \longrightarrow V$ called *the principal part of the field X*. In set-theretic notation,

$$X(v)(d) = v + d\xi(v)$$

In this case, we sometimes write $D_\xi(\beta)$ instead of $D_X(\beta)$.

Recall also that if $\beta : M \to V$ is any map into a Euclidean R-module, and X is a vector field on M, then the *directional derivative* $D_X(\beta)$ of β along X is the composite

$$M \xrightarrow{X} M^D \xrightarrow{\beta^D} V^D \to V,$$

where the last map is principal part formation. Using function theoretic notation, $D_X(\beta)$ is characterized by validity of the equation

$$\beta(X(m,d)) = \beta(m) + d \cdot D_X(\beta)(m),$$

for all $d \in D$, $m \in M$.

In the particular case $M = \tilde{R}$ and $V = R$, we have that the canonical map $\tilde{R} \times R \longrightarrow \tilde{R}^D$ is an isomorphism and one checks easily that the directional derivative takes the simpler form

$$D_X(\beta)(x) = \xi(x)\frac{d\beta}{dx}(x)$$

Proposition 2.0.10 *Assume that X_1, X_2 are vector fields on M_1, M_2, respectively, and that $H : M_1 \to M_2$ is a homomorphism (i.e., it preserves the D-action). Let V be a Euclidean R-module. Then for any $u : M_2 \to V$,*

$$D_{X_1}(u \circ H) = D_{X_2}(u) \circ H.$$

Proof. This is a straightforward computation:

$$u(X_2(H(m), d)) = u(H(m)) + d \cdot D_{X_2}(H(m));$$

on the other hand

$$u(X_2(H(m), d)) = u(H(X_1(m, d))) = u(H(m)) + d \cdot D_{X_1}(u \circ H)(m).$$

By comparing these two expressions we obtain the conclusion of the Proposition.

We now consider solutions $F : \tilde{R} \times V \to V$ for such vector fields, so equation (1) holds: $X_d \circ F_t = F_{t+d}$. In terms of principal parts, this equation may be rewritten as

$$\dot{F}_t(v) = \xi(F_t(v)).$$

Similarly, equation (2) may be written as

$$\dot{F}_t = D_\xi(F_t). \tag{6}$$

Using directional derivatives, we can give a more familiar expression to the vector field (1ODE) Y^X considered above on the object N^M, when the base N is a microlinear Euclidean R-module V, and the exponent M is microlinear. In fact, letting η be the principal part of the vector field Y on $N = V$, we have, for $u \in V^M$, $m \in M$, $d \in D$ (recall that $(X_d)^{-1} = X_{-d}$)

$$\begin{aligned}
(Y^X)_d(u)(m) &= Y_d \circ u \circ X_{-d}(m) \\
&= u((X_{-d}(m)) + d \cdot \eta(u(X_{-d}(m))) \\
&= u(m) - d \cdot D_X(u)(m) + d \cdot \eta(u(m)) \\
&= u(m) + d \cdot [-D_X(u)(m) + \eta(u(m))]
\end{aligned}$$

(at the third equality sign, a cancellation of $d \cdot d$ took place in the last term)

In other words, the principal part of Y^X is $\theta : V^M \longrightarrow V^M$ given by

$$\theta(u)(m) = \eta(u(m)) - D_X(u)(m).$$

NB. We recall that the 1ODE corresponding to a vector field X on a Euclidean R-module V may be written as $\dot{x} = \xi(x)$ where ξ is the principal part of X. This notation in fact may be justified by the following considerations: let $x : \tilde{R} \times V \longrightarrow V$ be a formal solution of X (which exists provided that V is microlinear). Therefore, $x(t+d, v) = X(d, x(t, v)) = x(t, v) + d\xi(v)$. This means that the derivative with respect to t, ("time") is given by $\dot{x}(t, v) = \xi(x(v, t))$. In other words, x is the solution of the 1ODE $\dot{x} = \xi(x)$ with initial condition $x(0, v) = X(0, v) = v$.

In these terms, the above equation may be rewritten (leaving out the m, and modulo some obvious abuse of notation) as

$$\dot{u} = \eta(u) - D_X(u),$$

or

$$\frac{\partial u}{\partial t} + D_X(u) = \eta(u).$$

This is a PDE of first order "in time". Thus, the exponential of two 1ODE's is a 1PDE.

In the particular case that $M = \tilde{R}$ and $N = R$, we can give this PDE, once again, a more familiar presentation. Indeed, assume that $U : \tilde{R} \times R^{\tilde{R}} \longrightarrow R^{\tilde{R}}$ is the formal solution of Y^X. Thus, $U(t+d, v) = Y_d^X(U(t.v)$ and this implies that

$$\frac{\partial U}{\partial t}(t.v) + D_X(U(v, t)) = \eta(U(v, t))$$

On the other hand,

$$D_X(U(t, v))(x) = \xi(x) \frac{dU(t, v)}{dx}(x)$$

Let v_0 a function thought as the initial condition. Then $u(t, x) = U(t, v_0)(x)$ is the solution of the partial differential equation

$$\frac{\partial u}{\partial t} + \xi(x) \frac{\partial u}{\partial x} = \eta(u)$$

with initial condition v_0.

We now prove (a form of the) chain rule. We need some preliminaries.

For any object N, let us consider its "zero vector field" Z , i.e., Z_d is the identity map on N, for all d. For a vector field X on an object M, we then also have the "vertical" vector field $Z \times X$ on $N \times M$.

If we have a complete solution $F : \tilde{R} \times M \to M$ of a vector field X on M, we may consider the map $\overline{F} : \tilde{R} \times M \to \tilde{R} \times M$ given by $(t, m) \mapsto (t, F(t, m))$

Proposition 2.0.11 *The map $\overline{F}$ thus described is an automorphism of the vector field $Z \times X$ on $\tilde{R} \times M$.*

Proof. By a straightforward diagram chase, one sees that this is a restatement of condition (3).

The following may be seen as a generalization of (6), and is a form of the chain rule. We consider a vector field X on M, with solution $F : \tilde{R} \times M \to M$. Let $U : \tilde{R} \times M \to V$ be any function with values in a Euclidean R-module.

Proposition 2.0.12 *Under these circumstances, we have*

$$\frac{\partial}{\partial t} U(t, F_t(m)) = \frac{\partial U}{\partial t}(t, F_t(m)) + (D_{Z \times X} U)(t, F_t(m))$$

for all $t \in \tilde{R}$, $m \in M$.

Proof. Since F is a solution of X, $F_{t+d} = X_d \circ F_t$, and so for any $t, t' \in \tilde{R}$ $(Z \times X)_d(t', F_t(m)) = (t', F_{t+d}(m)$. Therefore, by definition of directional derivative,

$$U(t', F_{t+d}(m)) = U(t', F_t(m)) + d \cdot (D_{Z \times X} U)(t', F_t(m)).$$

Putting $t' = t + d$, we thus have

$$\begin{aligned} U(t+d, F_{t+d}(m)) &= U(t+d, F_t(m)) + d \cdot (D_{Z \times X} U)(t+d, F_t(m)) \\ &= U(t+d, F_t(m)) + d \cdot (D_{Z \times X} U)(t, F_t(m)) \end{aligned}$$

by a standard cancellation of two d's, after Taylor expansion. Expanding the first term, we may continue:

$$= U(t, F_t(m)) + d \cdot \frac{\partial U}{\partial t}(t, F_t(m)) + d \cdot (D_{Z \times X} U)(t, F_t(m)).$$

On the other hand,

$$U(t+d, F_{t+d}(m)) = U(t, F_t(m)) + d \cdot \frac{\partial}{\partial t} U(t, F_t(m));$$

comparing these two expressions gives the result.

By specializing to the case $M = V = R$, we obtain

$$\frac{\partial}{\partial t} U(t, F(t,m)) = \frac{\partial U}{\partial t}(t, F(t,m)) + \frac{\partial F}{\partial t}(t,m) \frac{\partial U}{\partial x}(t, F(t,m))$$

which is clearly a more familiar form of the chain rule.

The method of change of variables has been used extensively to solve differential equations. We shall prove that our method for solving the exponential differential equation Y^X, where X is an integrable vector field on M, Y an integrable vector field on a Euclidean R-module, and where $\tilde{R}$ is symmetric with respect to the origin (if $t \in \tilde{R}$, then $-t \in \tilde{R}$), may be seen as an application of the method of change of variables. We let $\eta : V \to V$ denote the principal part of Y, as before. Let $F : \tilde{R} \times M \to M$ be the assumed solution of X, and let $\overline{F} : \tilde{R} \times M \to \tilde{R} \times M$ be the map

$$\overline{F}(t,m) = (t, F(-t,m))$$

Then $\overline{F}$ (which represents the change of variables $\tau = t$, $\mu = F(-t,m)$) is invertible.

Theorem 2.0.13 *("Change of variables"). If $u : \tilde{R} \times M \to V$ is a particular solution of Y^X, or, equivalently, of*

$$\frac{\partial u}{\partial t} + D_X(u) = \eta(u), \tag{7}$$

then the unique map $U : \tilde{R} \times M \to V$ given as the composite

$$\tilde{R} \times M \overset{(\overline{F})^{-1}}{\to} \tilde{R} \times M \overset{u}{\to} V$$

is a particular solution of Y^Z, or, equivalently, of

$$\frac{\partial U}{\partial t} = \eta(U), \tag{8}$$

and vice versa.

Proof. Since $u(t,m) = U(t, F_{-t}(m))$, we have

$$\frac{\partial u}{\partial t}(t,m) = \frac{\partial}{\partial t}U(t, F_{-t}(m)) = \frac{\partial U}{\partial t}(t, F_{-t}(m)) - D_{Z\times X}U(t, F_{-t}(m)),$$

by the chain rule, Proposition 2.0.12. On the other hand, $\overline{F}$ is an automorphism of the vector field $Z\times X$, by Proposition 2.0.11, and so, by construction of $\overline{F}$ and Proposition 2.0.10,

$$D_{Z\times X}(u) = D_{Z\times X}(U \circ \overline{F}) = (D_{Z\times X}U) \circ \overline{F}.$$

Therefore,

$$0 = \frac{\partial u}{\partial t} + D_{Z\times X}(u) - g(u)$$

$$= \frac{\partial U}{\partial t}(t,\mu) - D_{Z\times X}(U)(t,\mu) + D_{Z\times X}(U)(t,\mu) - \eta(U(t,\mu)),$$

where $\mu = F_{-t}(m)$, i.e., U is solution of

$$\frac{\partial U}{\partial t} = \eta(U),$$

proving the theorem (the vice versa part follows because $\overline{F}$ is invertible).

Corollary 2.0.14 (Uniqueness of solutions of the exponential) *Let X be a vector field on M and let Y be a vector field on an Euclidean module V. Assume that $F : \tilde{R} \times M \longrightarrow M$ is a complete solution of X. If Y has the uniqueness property for particular solutions, then so does Y^X.*

Proof: We let η be the principal part of Y. Let $u, w : \tilde{R} \times M \longrightarrow V$ be (exponential adjoints of) particular solutions of Y^X such that $u(0,m) = v(0,m)$. For each $m \in M$, define $U(t) = u(t, F(t,m)$ and $W(t) = w(t, F(t,m))$. By the change of variables theorem, both U and W satisfy $\dot{y} = \eta(y)$, i.e. they are particular solutions of Y with the same initial value. By uniqueness of particular solutions of Y, $U = W$. But m is arbitrary, i.e., $u(t, F(t,m)) = w(t, F(t,m))$ for every t and every m. Since $F(t,-)$ is bijective, this shows that $u = v$.

Corollary 2.0.15 (Uniqueness of the solution by conjugation) *Assume the hypothesis of the previous corollary. If Y has a complete solution G, then the solution of Y^X obtained by conjugation from F and G is the only complete solution of this exponential vector field.*

Some examples

Our method of solving exponentials of 1ODE's by conjugation of their solutions gives a straightforward algorithm to solve the resulting 1PDE. In fact, the solution of

$$\partial u/\partial t + \xi(x)\partial u/\partial x = \eta(u)$$

is simply

$$u(t,x) = G(t, v(F(-t,x)))$$

where v is an arbitrary function (the initial value), F and G are the solutions of the equations $\dot{x} = \xi(x)$ and $\dot{y} = \eta(y)$, respectively. This is a consequence of the general method in section 2, but we will check this in ordinary calculus. This is straightforward, once that we observe the following

Proposition 2.0.16 *Let* $f : \tilde{R} \times R \longrightarrow R$ *be a complete solution of the 1ODE* $\dot{y} = \xi(y)$. *Then*

$$\xi(f(t,x)) = \xi(x)(\partial f/\partial x)(t,x)$$

Proof: Let ln_ξ be a primitive of $1/\xi$. If $y(t)$ is a solution of $\dot{y} = \xi(y)$, then $ln_\xi(y(t)) = t + C$. Therefore,

$$ln_\xi(f(t,x)) = t + ln_\xi(x)$$

Taking the partial derivative with respect to x,

$$(\partial f/\partial x)(t,x)/\xi(f(t,x)) = 1/\xi(x)$$

which is the required formula.

NB. Since $\xi(f(t,x) = \dot{f}(t,x)$, we have shown that f is a (complete) solution of the 1PDE

$$\partial u/\partial t - \xi(x)\partial u/\partial x = 0$$

It is easy to show that, conversely, any solution of this PDE is a solution of $\dot{y} = \xi(y)$.

Example 1 ('the transport equation')

$$\partial u/\partial t + \partial u/\partial x = 0$$

Here, $\xi(x) = 1$ and $\eta(y) = 0$. The complete solution of $\dot{y} = y$ is clearly $F(x,t) = x + t$, which is globally defined. On the other hand, the complete solution of $\dot{y} = 0$ is $G(t,x) = x$ (also globally defined). Hence

$$H(t,x) = G(t, v(F(-t,x))) = v(x-t)$$

is the only (globally defined) complete solution of the PDE.

Example 2

$$\partial u/\partial t + x\partial u/\partial x = u$$

In this case, $\xi(x) = x$ and $\eta(y) = y$ and their complete solutions are the same, namely $F(t,x) = G(t,x) = xe^t$ (globally defined). Therefore,

$$H(t,x) = G(t, v(F(-t,x))) = v(xe^{-t})e^t$$

is the (globally defined) complete solution of the PDE.

Example 3 Let D be the set of elements of square zero in R, as usual. It carries a vector field, namely the map $e : D \times D \to D$ given by $(d, \delta) \mapsto (1 + d) \cdot \delta$. It is easy to see that this vector field is integrable, with complete solution $E : R \times D \to D$ given by $(t, \delta) \mapsto e^t \cdot \delta$. Now consider the tangent vector bundle M^D on M. The zero vector field Z on M is certainly integrable, and so we have by the theorem a complete integral for the vector field Z^e on the tangent bundle. We describe the integral explicitly (this then also describes the vector field, by restriction): it is the map $R \times M^D \to M^D$ given by $(t, \beta) \mapsto [d \mapsto \beta(e^{-t} \cdot d)]$.— The vector field on M^D obtained this way is, except for the sign, the *Liouville vector field*, cf. [2], IX.2.

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