Embedding manifolds with boundary in smooth toposes

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Résumé
Nous construisons un plongement plein et fidèle de la catégorie des variétés à bord dans certains topos "lisses" dont le topos du Cahiers [2] et le topos $\mathcal{F}$ d'idéaux fermés [9]. Nous démontrons que le plongement préserve les produits d'une variété à bord avec une variété sans bord et les recouvrements ouverts. De plus, il envoie des prolongations de variétés par algèbres de Weil dans des exponentielles ayant des structures infinitésimales comme exposants. Notre outil principal est l'opération de "doubler" une variété à bord pour obtenir une variété sans bord. Cet article est une réécriture, avec quelques améliorations, de [11].

In [2], E.Dubuc introduced the notion of a well adapted model of SDG (Synthetic Differential Geometry), and built one such, the *Cahiers* topos as is usually called, showing that SDG was applicable to the study of classical smooth manifolds without boundary. (See also [5].) These models are, in fact, adapted to differential calculus, but the development of integral calculus in the context of SDG (cf [6]) requires a finer notion of adaptation. In fact, integral calculus requires the notion of closed interval, a notion modelled rather by manifolds with boundary such as $\mathbb{R}_{\geq 0}$ and even manifolds with corners. The aim of this paper is to exhibit a fully faithful embedding of the category $\mathcal{M}_\partial$ of manifolds with boundary in the *Cahiers* topos as well as other "smooth" toposes (to be defined later on.)

This paper is a re-working of and hopefully an improvement on [11]. The exact relation to that paper is explicitly stated in the remark just before theorem 2.5.

We are very grateful to Anders Kock for his careful reading of two previous versions of this paper that saved us from error and awkward formulations on more than one occasion.

1 Generalities on manifolds with boundary

Along with the space $\mathbb{R}^n$, the simplest manifold of dimension n, there are certain subspaces which are almost equally important, but fail to be manifolds. Possibly the simplest such is $\mathbb{H} =$ the non-negative reals. More generally, let $\mathbb{H}^n = \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$. (Thus $\mathbb{H} = \mathbb{H}^1$). The space $\mathbb{H}^n$ has an obvious boundary $\partial\mathbb{H}^n = \{0\} \times \mathbb{R}^{n-1}$ which can be easily visualized in lower dimensions: a point $n = 1$, a line for $n = 2$, a plane for $n = 3$, etc. Taking this space as a model, we define the notion of a manifold with boundary, following [10].

A (topological) *manifold with boundary* B is a Hausdorff topological space with the property that every point has a neighborhood homeomorphic to either $\mathbb{R}^n$ or $\mathbb{H}^n = \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$. We call n the *dimension* of B. If b has a neighborhood homeomorphic to $\mathbb{H}^n$ by a homeomorphism h such that $h(x) \in \partial\mathbb{H}^n$, we say that b is a *boundary point*. The set of of all such points, ∂B is called the *boundary* of B. We define the *interior* of B, $int(B)$ to be the set $int(B) = B \setminus \partial B$.

For these definitions to make sense, we need to prove several things, e.g. that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ whenever $m \neq n$, that boundary points are independent of the homeomorphisms, etc. These are technical matters, depending essentially on Brouwer theorem on invariance of dimension. The reader may consult [10].

A C^∞ manifold (with boundary) B of dimension n consists of a topological manifold with boundary of dimension n together with a differentiable structure $\mathcal{D}$. A *differentiable structure* is a collection of couples (U, h) such that the U's are open sets of B and h a homeomorphism of U with either $\mathbb{R}^n$ or $\mathbb{H}^n$ satisfying the following properties:

(1) The U's cover B

(2) If (U_1, h_1) and (U_2, h_2) belong to $\mathcal{D}$, then the map

$$h_1 h_2^{-1} : h_2(U_1 \cap U_2) \longrightarrow \mathbb{R}^n$$

is C^∞.

(3) The collection $\mathcal{D}$ is maximal with respect to property (2).

In (2) we still need to define C^∞ functions with domain $\mathbb{H}^n$. We say that $f : \mathbb{H}^n \longrightarrow \mathbb{R}^n$ is C^∞ at a point $p \in \partial\mathbb{H}^n$ if there is an open neighborhood V of p in $\mathbb{R}^n$ and a C^∞ function $g : V \longrightarrow \mathbb{R}^n$ such that g and f coincide on $V \cap \mathbb{H}^n$. A function $f : \mathbb{H}^n \longrightarrow \mathbb{R}^n$ is C^∞ if it is C^∞ at every point.

In practice, to define such a manifold it suffices to define a collection (U, h) satisfying (1) and (2), since we can always extend it uniquely to a differentiable structure.

Ordinary manifolds are those whose boundary is empty. From now on, we consider only C^∞ manifolds (with boundary). These we call simply "manifolds with boundary".

In [10], the notion of morphism between manifolds with boundary is also defined. We let $\mathcal{M}_\partial$ the category of manifolds with boundary and $\mathcal{M}$ the category of manifolds without boundary.

Let B be a manifold with boundary. We say that a manifold with boundary C is a *submanifold* of B if, as sets, C is included in B and the inclusion map is smooth. A *closed submanifold* of B is a submanifold which is a closed subspace of B (as a topological space).

Our main tool to study manifolds with boundary is the operation of "doubling" a manifold with boundary B to obtain its double $D(B)$, a manifold without boundary. Properties of manifolds with boundary will then be reduced to the corresponding ones for manifolds without boundary.

If B is a manifold with boundary, we let $D(B)$ the manifold without boundary obtained by taking two disjoint copies of B and identifying their boundaries (cf. [10] for a precise description.) This construction, although functorial in the category of topological manifolds with boundary, it is not functorial in the category of smooth manifolds with boundary. There is no canonical differential structure on $D(M)$ and the definition of a differentiable structure depends on the choice of a product neighbourhood of ∂B. Nevertheless, structures resulting from different choices are diffeomorphic. The construction, however, has some good properties:

Proposition 1.1

(i) $D(B)$ *is a manifold without boundary*

(ii) B *is canonically embedded in* $D(B)$ *as a closed subset*

(iii) If $H \subset B$ *is open,* $D(H) \cap B = H$

(iv) If $H \subset B$ *is open,* $D(H)$ *is an open subset of* $D(B)$

Proof: (i) can be found in [10]. (ii) is straightforward. We turn to (iii). Let $p \in H \cap D(B)$. The there is an open $H' \hookrightarrow B$ and a $\phi : H' \simeq \mathbb{H}^n$, sending p into $x \in \partial \mathbb{H}^n$. Let $U = \phi(H \cap H')$. Then $x \in U \subseteq \mathbb{H}^n$. Since U is open in $\mathbb{H}^n$ we may find, using stereographic projections, a $V \ni x$ open in U and a diffeomorphism $\psi : V \simeq \mathbb{H}^n$ leaving x fixed. But $\phi^{-1}(V)$ contains p, is an open subset of H and is diffeomorphic to $\mathbb{H}^n$ by a diffeomorphism (namely $\psi \circ \phi$) that sends p into $x \in \partial \mathbb{H}^n$. This concludes the proof. (iv) is an immediate corollary of (iii).

Weil prolongations ([12], see also [2]) can be extended to manifolds with boundary. If B is a manifold with boundary and W is a Weil algebra, we define a manifold with boundary ${}^W B$=manifold of points W*-close of* B to be ${}^W B = C^\infty - alg(C^\infty(B),\ W)$. To show that it is a manifold, some preliminary work is required.

Notice that there is a canonical map $\rho : {}^W B \longrightarrow {}^{\mathbb{R}} B$ defined as the composite of ϕ with the canonical map $\pi_0 : W \longrightarrow \mathbb{R}$. But ${}^{\mathbb{R}} B = C^\infty - alg(C^\infty(B), \mathbb{R})$ may be identified with B. Indeed, embed B as a closed set in some $\mathbb{R}^n$ and let $\tau : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(B)$ be the canonical quotient (Whitney's theorem!). Take $\phi : C^\infty(B) \longrightarrow \mathbb{R}$. By composing it with τ we obtain a map $\tilde{\phi} : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$. But this map is just $ev(p)$ for some $p \in \mathbb{R}^n$. Using smooth Urysohn ([3], page 56), one can easily prove that $ev(p)$ factors through τ iff $p \in B$. Thus, $p \in B$, showing that ρ may be viewed as a map $\rho : {}^W B \longrightarrow B$.

To show that ${}^W B$ is a manifold we proceed as follows: since B is such a manifold, for each $b \in B$, there is an open neighbourhood H_b of b which is diffeomorphic to either $\mathbb{R}^n$ or $\mathbb{H}^n$. The family $\{H_b\}_{b\in B}$ is an open cover of B. We call such a cover a *Euclidean cover* of B. The "double" of this cover, $\{D(H_b)\}_{b\in B}$ is an open cover of $D(B)$. (Cf. 1.1). . Notice that

$$D(H_b) \simeq \begin{cases} \mathbb{R}^n + \mathbb{R}^n & \text{if } H_b \cap \partial B = \emptyset \\ \mathbb{R}^n & \text{otherwise} \end{cases}$$

From [2], we know that $\{{}^W D(H_b)\}_{b\in B}$ is an open cover of ${}^W D(H)$. We claim that $\{{}^W H_b\}_{b\in B}$ is an open cover of ${}^W B$. Since ${}^W H_b$ is homeomorphic either to ${}^W\mathbb{R}^n \simeq \mathbb{R}^{nk}$ or ${}^W\mathbb{H}^n \simeq \mathbb{H}^{nk}$, where k=dimension of W (as a vector space over $\mathbb{R}$) (cf. lemma 1.5 below), this will show that ${}^W B$ is a manifold (with boundary) of dimension nk.

This is a consequence of the following series of lemmas and corollaries

Lemma 1.2 *Let $H \hookrightarrow B$ be open (B a manifold with boundary). Then the morphism ${}^W H \longrightarrow {}^W B$ is monic.*

Proof: Unravelling the definitions this lemma says that whenever we have a commutative diagram

$$\begin{array}{ccc} C^\infty(B) & & \\ \downarrow{\scriptstyle \rho} & \searrow^{\phi} & \\ C^\infty(H) & \overset{\phi_1}{\underset{\phi_2}{\rightrightarrows}} & W \end{array}$$

then $\phi_1 = \phi_2$.

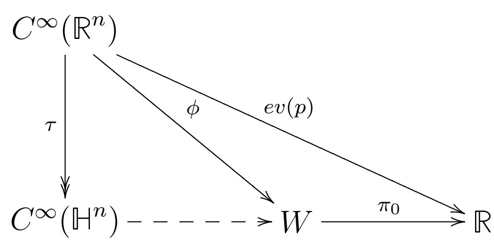
Since $D(H)$ is open in $D(B)$, the map ${}^W D(H) \longrightarrow {}^W D(B)$ is monic (Cf. [2]) and this means that in the outer diagram of

$$\begin{array}{ccccc} C^\infty(D(B)) & \xrightarrow{\tau'} & C^\infty(B) & & \\ \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} & \searrow^{\phi} & \\ C^\infty(D(H)) & \xrightarrow[\tau]{} & C^\infty(H) & \overset{\phi_1}{\underset{\phi_2}{\rightrightarrows}} & W \end{array}$$

$\phi_1 \circ \tau = \phi_2 \circ \tau$. Since τ is a quotient, $\phi_1 = \phi_2$.

The same is true if H is closed in B, since $\tau : C^\infty(B) \longrightarrow C^\infty(H)$ is a quotient.

Lemma 1.3 *Let W be a Weil algebra and $\pi_0 : W \longrightarrow \mathbb{R}$ the canonical map. Let $\tau : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{H}^n)$ be the canonical quotient. A morphism $\phi : C^\infty(\mathbb{R}^n) \longrightarrow W$ of C^∞-algebras factors as shown iff the unique $p \in \mathbb{R}^n$ such that $\pi_0 \circ \phi = ev(p)$ belongs to $\mathbb{H}^n$*



Proof: Assume that $p \in \mathbb{H}^n$. We have to show that ϕ factors through τ or, equivalently, that $\phi(f) = 0$ whenever $f|_{\mathbb{H}^n} = 0$. Since W is a Weil algebra, there is a $k \in \mathbb{N}$ such that $m_W^{k+1} = 0$, where m_W is the maximal ideal of W.

According to the Taylor development of f around p with Hadamard's remainder, there are functions g_α with $|\alpha| = k+1$ (not necessarily unique) such that

$$f(x) = \Sigma_{|\alpha|=0}^{|\alpha|=k} 1/\alpha!(x_1 - p_1)^{\alpha_1} \ldots (x_n - p_n)^{\alpha_n}\, \partial^{|\alpha|} f(p)/\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$
$$+ \textstyle\sum_{|\alpha|=k+1} (x_1 - p_1)^{\alpha_1} \ldots (x_n - p_n)^{\alpha_n} g_\alpha(x,p)$$

As usual, we have used $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$ whenever $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Since f vanishes on $\mathbb{H}^n$, it is flat in p and so the first sum vanishes, i.e.,

$$f(x) = \textstyle\sum_{|\alpha|=k+1} (x_1 - p_1)^{\alpha_1} \ldots (x_n - p_n)^{\alpha_n} g_\alpha(x,p)$$

Applying ϕ

$$\phi(f) = \textstyle\sum_{|\alpha|=k+1} w_1^{\alpha_1} \ldots w_n^{\alpha_n} \phi(g_\alpha(x,p))$$

where $w_i = \phi(x_i - p_i)$.

But $w_i \in m_W$, since $\pi_0 \circ \phi(x_i - p_i) = ev(p)(x_i - p_i) = 0$. Therefore this term vanishes, i.e., $\phi(f) = 0$.

The proof in the other direction: if $\overline{\phi}$ is the quotient of ϕ, $\pi_0 \circ \overline{\phi} = ev(q)$ for a unique $q \in B$. Since functions separate points, $p = q$.

Corollary 1.4 *The square*

$$\begin{array}{ccc} {}^W\mathbb{H}^n & \hookrightarrow & {}^W\mathbb{R}^n \\ \downarrow & & \downarrow \\ \mathbb{H}^n & \hookrightarrow & \mathbb{R}^n \end{array}$$

is a pull-back.

Proof: Immediate by lemma 1.3.

Lemma 1.5

$${}^W\mathbb{H}^n \simeq \mathbb{H}^{nk}$$

where k is the linear dimension of W over $\mathbb{R}$.

Proof: This follows from ${}^W\mathbb{R}^n \simeq \mathbb{R}^{nk}$ and the fact that the square

$$\begin{array}{ccc} \mathbb{H}^{nk} & \hookrightarrow & \mathbb{R}^{nk} \\ \downarrow & & \downarrow \\ \mathbb{H}^n & \hookrightarrow & \mathbb{R}^n \end{array}$$

is a pull-back which, in turn, is a consequence of the fact that any square

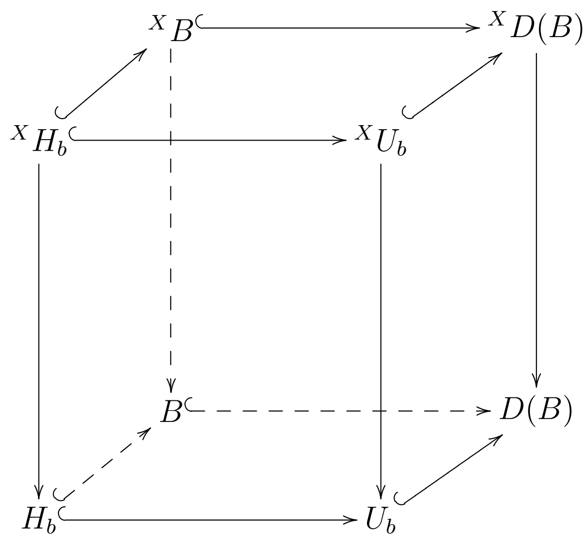
$$\begin{array}{ccc} C \times A & \hookrightarrow & C \times B \\ \downarrow & & \downarrow \\ A & \hookrightarrow & B \end{array}$$

is obviously a pull-back.

Lemma 1.6 *If $\{H_b \hookrightarrow B\}_{b \in B}$ is a Euclidean cover, then*

$$\bigcup_{b \in B} {}^W H_b = {}^W B.$$

Proof: Chase elements in the diagram



recalling that $\bigcup_{b \in B} {}^W D(H_b) = {}^W D(B)$.

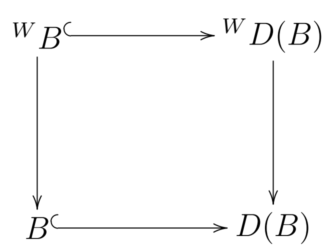
Corollary 1.7 *If $\{H_\alpha \hookrightarrow B\}_\alpha$ is any open cover of B,*

$$\bigcup_\alpha {}^W H_\alpha = {}^W B$$

Proof: Cover each H_α by a Euclidean cover.

This finishes the proof that ${}^W B$ is a manifold (with boundary).

Corollary 1.8 *The following diagram is a pull-back*



Proposition 1.9 *If W is a Weil algebra, then ${}^W B \hookrightarrow {}^W D(B)$ is closed.*

Proof: Use Euclidean covers and the fact that "being closed" is a property of local character. In details: let $\{H_b | b \in B\}_{b \in B}$ be a Euclidean cover of B. Then we have commutative diagrams, according to whether $H_b \cap \partial B = \emptyset$ or not

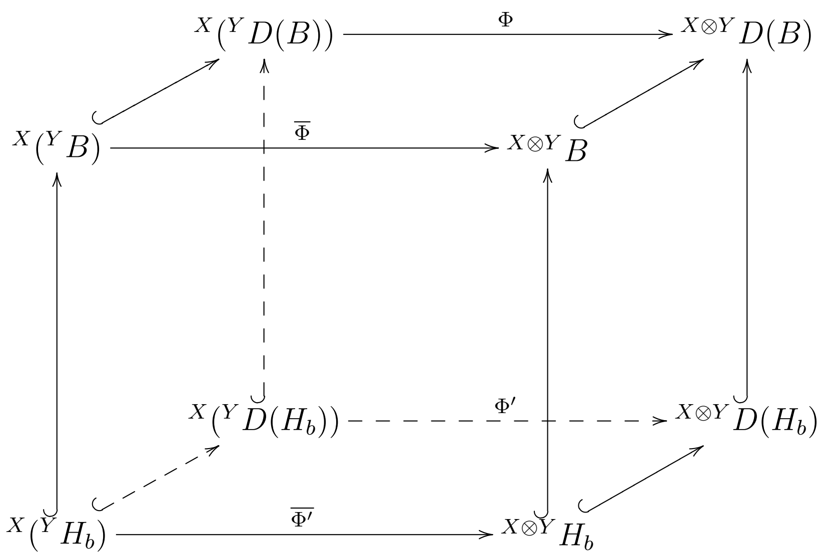
$$
\begin{array}{ccc}
{}^W H_b & \hookrightarrow & {}^W D(H_b) \\
\simeq \uparrow & & \uparrow \simeq \\
\mathbb{R}^{nk} & \hookrightarrow & \mathbb{R}^{nk} + \mathbb{R}^{nk}
\end{array}
\qquad\qquad
\begin{array}{ccc}
{}^W H_b & \hookrightarrow & {}^W D(H_b) \\
\simeq \uparrow & & \uparrow \simeq \\
\mathbb{H}^{nk} & \hookrightarrow & \mathbb{R}^{nk}
\end{array}
$$

Notice that in the first, we have used that the two copies of $\mathbb{R}^n$ form an open cover of $\mathbb{R}^n + \mathbb{R}^n$. Furthermore, $\mathbb{R}^n \hookrightarrow \mathbb{R}^n + \mathbb{R}^n$ and $\mathbb{H}^{nk} \hookrightarrow \mathbb{R}^{nk}$ are closed. Hence, in both cases, ${}^W H_b \hookrightarrow {}^W D(H_b)$ is closed for each $b \in B$. Let $\phi \in {}^W D(B) \setminus {}^W B$. Then there is a $b \in {}^W D(H_b) \setminus {}^W H_b$. The image of ϕ under the above diffeomorphisms belongs to either $\mathbb{R}^{nk} + \mathbb{R}^{nk} \setminus \mathbb{R}^{nk}$ or $\mathbb{R}^{nk} \setminus \mathbb{H}^{nk}$ both of which are open (the first in $\mathbb{R}^{nk} + \mathbb{R}^{nk}$, the other in $\mathbb{R}^{nk}$). Hence, their images under the diffeomorphism are open in ${}^W D(H_b)$ and disjoint from ${}^W H_b$. This concludes the proof.

In [2], a canonical map $\Phi : {}^X({}^Y D(B)) \longrightarrow {}^{X \otimes Y} D(B)$ is constructed and shown to be a bijection. The same formula defining Φ defines a canonical map: $\overline{\Phi} : {}^X({}^Y B) \longrightarrow {}^{X \otimes Y} B$ in such a way that the top of the next diagram commutes

Proposition 1.10 *The map $\overline{\Phi}$ is a bijection.*

Proof: Take a Euclidean cover $\{H_b | b \in B\}_{b \in B}$ of B and chase the diagram



noticing that Φ and Φ' are bijections ([2]).

The following plays a key role in the main theorem (theorem 2.5). For the notions of near-point determined algebra, closed ideal and their relations, see e.g. [9] page 44.

Proposition 1.11 *Let W be a Weil algebra and A a near-point determined algebra. Then there are natural (in A) canonical bijections*

$$\frac{C^\infty({}^W\mathbb{R}^n) \longrightarrow A}{C^\infty(\mathbb{R}^n) \longrightarrow A\otimes W} \qquad \frac{C^\infty({}^W\mathbb{H}^n) \longrightarrow A}{C^\infty(\mathbb{H}^n) \longrightarrow A\otimes W}$$

Proof: Let us prove the first. Since ${}^W\mathbb{R}^n = C^\infty - alg(C^\infty(\mathbb{R}^n), W)$, we have ${}^W\mathbb{R}^n = W^n$.

Let $\{1, w_1, \ldots, w_{k-1}\}$ be a linear basis of W, i.e., $W = \mathbb{R}[w_1, \ldots, w_{k-1}] = \mathbb{R}^k$. Hence $W^n = (\mathbb{R}^k)^n$, $A \otimes W = A[w_1, \ldots, w_{k-1}]$ and $C^\infty({}^W\mathbb{R}^n) \otimes W = (C^\infty(\mathbb{R}^k)^n))^k$. With these identifications, we have to show the following equivalence

$$\frac{\phi : C^\infty((\mathbb{R}^k)^n) \longrightarrow A}{\psi : C^\infty(\mathbb{R}^n) \longrightarrow A[w_1, \ldots, w_{k-1}]}$$

Following [2], we define

$$\psi(\pi_i) = \phi(\pi_1\pi_i^k) + w_1\phi(\pi_2\pi_i^k) + \ldots + w_{k-1}\phi(\pi_k\pi_i^k)$$

Notice that the $\pi_i\pi_j^k$ constitute a set of generators for $C^\infty((\mathbb{R}^n)^k)$. Then ϕ is completely determined by the nk elements $a_{ij} = \phi(\pi_i\pi_j^k)$. Similarly, ψ is completely determined by the n elements $\psi(\pi_i)$ of $A[w_1, \ldots, w_{k-1}]$. But each of these is, in turn, determined by the k elements $(a_{1i}, \ldots, a_{ki})$. Thus, ϕ and ψ are determined by the same nk elements of A. This concludes the proof.

For the second, we formulate first a lemma. For the definition of closed ideal, see e.g. [9] page 44.

Lemma 1.12 *Let I be a closed ideal in $C^\infty(\mathbb{R}^m)$ and let $a_i : \mathbb{R}^m \longrightarrow \mathbb{R}$ be smooth functions ($i = 1, \ldots, n$). The following are equivalent:*

1. *For every $\mu \in m_{\mathbb{H}^n}$, $\mu(a_1(x), \ldots, a_n(x)) \in I$*
2. *For every $\rho \in m_{\mathbb{H}}$, $\rho(a_1(x)) \in I$*
3. *$a_1(x_0) \geq 0$ for all $x_0 \in Z(I)$*

Proof:

1 implies 2: Let $\rho \in m_{\mathbb{H}}$. Define $\mu(u_1, \ldots, u_n) = \rho(u_1)$. Then $\rho(a_1(x)) = \mu(a_1(x), \ldots, a_n(x)) \in I$ since $\mu \in m_{\mathbb{H}^n}$.

2 implies 3: Since $\mathbb{H}$ is closed, we may pick ρ such that $\rho^{-1}(0) = \mathbb{H}$. (See e.g. [9] page 30). Then, obviously, $\rho(a_1(x_0)) = 0$. Hence $a_1(x_0) \geq 0$ (by choice of ρ.)

3 implies 1: Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ be defined by $f(x) = \mu(a_1(x), \ldots, a_n(x))$. The Taylor's development of f around x_0 (as a formal power series) is

$$T_{x_0}(f) = f(x_0) + \sum_\alpha (\partial^{|\alpha|} f / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n})(x_0) h_1^{\alpha_1} \cdots h_n^{\alpha_n}$$

with $h_i = x_i - (x_0)_i$. Let $\mu \in m_{\mathbb{H}^n}$ and $x_0 \in Z(I)$. Therefore $a_1(x_0) \geq 0$ and hence $f(x_0) = 0$. Furthermore, since all partial derivatives contain partial derivatives of μ at x_0 as factors, these terms are 0 (μ is flat at $(a_1(x_0), \ldots, a_n(x_0))$ and so $T_{x_0}(f) = 0$. A fortiori, $T_{x_0}(f) \in T_{x_0}(I)$ for every $x_0 \in Z(I)$ and so $f \in I$ by the very definition of I being closed.

If $A = C^\infty(\mathbb{R}^n)/I$, we shall write $A \Vdash (a_1, \ldots a_n) \in H^n$ as a shorthand for the statement "$\forall \mu \in m_{\mathbb{H}^n} \mu(a(x)) \in I$" where $a_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a function whose equivalence class modulo I is a_i ($i = 1, \ldots n$). Instead of $A \Vdash a \in H$ we sometimes write $A \Vdash a \geq 0$.

Corollary 1.13 *Let* $A = C^\infty(\mathbb{R}^m)/I$ *with* I *closed and* $\phi : C^\infty(\mathbb{R}^n) \longrightarrow A$. *Then* ϕ *factors as shown*

$$\begin{array}{ccc} C^\infty(\mathbb{R}^n) & & \\ \downarrow{\scriptstyle\tau} & \searrow{\scriptstyle\phi} & \\ C^\infty(\mathbb{H}^n) & \dashrightarrow & A \end{array}$$

iff $A \Vdash \phi(\pi_1) \geq 0$.

Proof: The map ϕ factors as indicated iff for every $\mu \in C^\infty(\mathbb{R}^n)$ if $\tau(\mu) = 0$, then $\phi(\mu) = 0$, i.e., if for every $\mu \in m_{\mathbb{H}^n}$, $\mu(a_1(x), \ldots a_n(x) \in I$, where $a_i : \mathbb{R}^m \longrightarrow \mathbb{R}$ are functions whose equivalence classes (modulo I) are $\phi(\pi_i)$ $(i = 1, \ldots, n.)$ By the lemma, this is equivalent to say that $A \Vdash \phi(\pi_1) \geq 0$.

To complete the proof of the second formula, we recall (cf. [9]) that nilpotent elements are ≥ 0 and the sum of two elements ≥ 0 is again ≥ 0. Thus, returning to the proof of the first (formula) $A[w_1, \ldots, w_k] \Vdash \psi(\pi_1) \geq 0$ iff $A \Vdash \phi(\pi_1 \pi_1^k) \geq 0$. Apply now the corollary.

2 Smooth toposes

Although several notions of "smooth toposes" besides the *Cahiers* topos have been introduced in the literature (say [9] I.4) we shall reserve this terminology for a particular class, and warn the reader that it does not cover all of them.

A *smooth topos* is a category of sheaves on a co-site $\mathbb{E}$ of C^∞-rings having the following properties

$$\begin{cases} \text{Rings of manifolds (without boundary)} \;\; C^\infty(M) \;\; \text{belong to} \;\; \mathbb{E} \\ \text{Weil algebras belong to } \mathbb{E} \\ \text{All rings of } \;\mathbb{E}\; \text{ are near-point determined} \\ \text{Representables are sheaves} \\ \{C^\infty(M) \longrightarrow C^\infty(U_\alpha)\}_\alpha \;\; \text{is a cocovering family,} \end{cases}$$

In the last property, we assume that $\{U_\alpha \hookrightarrow M\}_\alpha$ is an open cover of $M \in \mathcal{M}$.

For several purposes, it is more natural to consider the category $\mathbb{L}$ of loci, defined to be $\mathbb{E}^{op}$, the opposite category of $\mathbb{E}$ rather than $\mathbb{E}$ itself. Its objects are the same as those of $\mathbb{E}$ but the arrows are reversed. If A and B are C^∞-algebras and $l(A)$ and $l(B)$ the same algebras considered as loci, we have, by definition, the equivalence

$$\frac{l(A)\longrightarrow l(B) \qquad \text{in } \mathbb{L}}{B\longrightarrow A \qquad \text{in } C^\infty - rings}$$

The cocovers of $\mathbb{E}$ become covers in $\mathbb{L}$ and this helps the intuition. We shall not distinguish between these categories notationally, but the context will indicate which one we have in mind. For instance, if we say that covers are preserved, it is clear that we are thinking of $\mathbb{L}$.

We let $\mathcal{E}$ to be the (Grothendieck) topos defined by the co-site $\mathbb{L}$.

Let us remark that the inclusion of Weil algebras in $\mathbb{E}$ guarantees that the generalized Kock-Lawvere axiom is valid in the topos $\mathcal{E}$. Thus, every smooth topos satisfies this axiom. The inclusion of those particular cocovers, on the other hand, will guarantee that open covers of manifolds without boundary will be preserved by the embedding in $\mathcal{E}$ to be described below.

A natural solution to the problem of embedding manifolds with boundary in a smooth topos is to represent $B \in \mathcal{M}_\partial$ from "outside": we define

$$\begin{cases} \mathcal{M}_\partial \longrightarrow \mathcal{E} \\ B \longmapsto a h_0^{C^\infty(B)} \end{cases}$$

where a is the associated sheaf and $h_0^{C^\infty(B)}$ is the restriction of the functor representable by $C^\infty(B)$ to $\mathbb{E}$, i.e., $h_0^{C^\infty(B)}(A) = C^\infty - rings(C^\infty(B), A)$.

As usual in these studies, it is very hard to work with the associated sheaf functor. Fortunately, we have the following

Proposition 2.1 *Let B be a manifold with boundary. Then the functor $h_0^{C^\infty(B)}$ is a sheaf in every smooth topos.*

Proof: Embed B as a closed subset of some $\mathbb{R}^n$. Let $\{A\longrightarrow A_\alpha\}_\alpha$ be a co-covering family and $\{\xi_\alpha : C^\infty(B)\longrightarrow A_\alpha\}_\alpha$ a compatible family. Then the family $\{\xi_\alpha \circ \tau\}_\alpha$ is still compatible, where $\tau : C^\infty(\mathbb{R}^n)\longrightarrow C^\infty(B)$ is the canonical quotient. Since the functor representable by $C^\infty(\mathbb{R}^n)$ is a sheaf, by definition of a smooth topos, there is a unique morphism $\theta' : C^\infty(\mathbb{R}^n)\longrightarrow A$ such that $\rho_\alpha \circ \theta' = \xi_\alpha \circ \tau$.

We claim that θ' factors through τ. Let $f \in C^\infty(\mathbb{R}^n)$ be such that $f|_B = 0$. Since $\rho_\alpha \circ \theta'$ factors through τ, $\rho_\alpha \circ \theta'(f) = 0$ for each α. But this clearly implies that $\theta'(f) = 0$. (Proof: define $\phi : C^\infty(\mathbb{R})\longrightarrow A$ by $\phi(id) = \theta'(f)$ and use the fact that the functor representable by $C^\infty(\mathbb{R})$ is a sheaf.)

So, we have thus the embedding of Weil algebras

$$\begin{cases} \mathbb{W}\longrightarrow\mathcal{E} \\ W \longmapsto l(W) = h^W \end{cases}$$

and the embedding of manifolds (with boundary)

$$\begin{cases} \mathcal{M}_\partial\longrightarrow\mathcal{E} \\ B \longmapsto s(B) = h^{C^\infty(B)} \end{cases}$$

Two objects of $\mathcal{E}$ are particularly important in what follows: $R = s(\mathbb{R})$ ("the reals") and $H = s(\mathbb{H})$ ("the non-negative reals").

In our main reference for sheaf semantics [9] , if A is a C^∞-algebra, $l(A)$ is used for A itself, but as an object of the opposite of the category of C^∞-algebras. This creates no problem, since Yoneda lemma allows us to identify $l(A)$ with h^A.

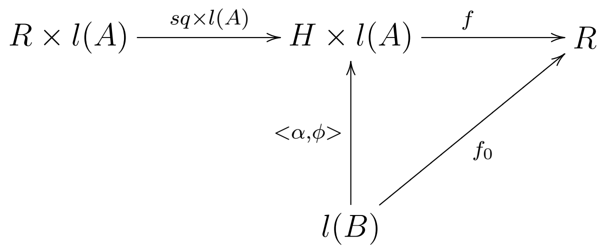
Lemma 2.2 *The map $sq : R\longrightarrow H$ in $\mathcal{E}$ defined by $sq(x) = x^2$ is dense in the sense that $R^{sq} : R^H\longrightarrow R^R$ is monic.*

Proof: Recall from [9] that the category of loci is the opposite of the category of finitely generated C^∞-algebras (i.e., of the form $A = C^\infty(\mathbb{R}^n/I$ where I is an ideal). If A is such an algebra, $l(A)$ is this algebra, but considered as a locus. In what follows, we identify $l(A)$ with $h^A \in \mathcal{E}$. Tis is possible since representables are sheaves in $\mathcal{E}$.

Recall from [9] that if A is a C^∞-algebra, an element f of R^H defined at the stage $l(A) = h^A$, is a map $f : H \times l(A)\longrightarrow R$. The composite $f \circ sq$ (defined at the same stage) is the map:

$$R \times l(A) \xrightarrow{sq\times l(A)} H \times l(A) \xrightarrow{f} R$$

It suffices to show (since R is a ring) that if $f \circ sq \times l(A) = 0$, then $f = 0$. Consider the commutative diagram with $< \alpha, \phi >$ arbitrary



It is enough to show that $f_0 = 0$, but since B is near-point determined, we may assume, without loss of generality, that B is a Weil algebra. We claim that $f_0 = f \circ < \alpha, \phi >= 0$.

In fact, either $\alpha(0) > 0$ and the claim follows at once, or $\alpha(0) = 0$. In this case, the claim follows from the following synthetic argument:

Lemma 2.3 *Let $f : H \longrightarrow R$ be such that $f(x^2) = 0$. Then $f|_{D_\infty} = 0$.*

Proof: We prove by induction that $\forall n \in N\ \exists f_n : H \longrightarrow R$ such that $f(x) = x^n f_n(x)$. There is nothing to prove for $n = 0$. Assume that this holds for n. Therefore, $0 = f(x^2) = x^{2n} f_n(x^2)$ and by repeated applications of the Lavendhomme principle ($xf(x) \equiv 0$ implies $f(x) \equiv 0$, cf. [7], page 25) $f_n(x^2) = 0$. In particular, $f_n(0) = 0$ and hence, by Hadamard's, there is $f_{n+1} : H \longrightarrow R$ such that $f_n(x) = x f_{n+1}(x)$. Hence $f(x) = x^n f_n(x) = x^{n+1} f_{n+1}(x)$, completing the induction. Let n be given. Since $f(x) = x^{n+1} f_{n+1}(x)$, we easily conclude that $f^n(0) = 0$.

Remark 2.4 The same argument generalizes to show that $sq_n : R^n \longrightarrow H^n$ is dense, where $H^n = \{(x_1, \ldots, x_n) \in R^n | x_1 \geq 0\}$ and $sq_n(x_1, \ldots, x_{n-1}, x_n) = (x_1^2, \ldots, x_{n-1}, x_n)$.

The following theorem is the main result of the paper. It improves [11] in the following respects:

1. It corrects several errors and provides detailed proofs that were only sketched.

2. It is more general in the sense that it covers e.g, not only the *Cahiers* topos, but the topos $\mathcal{F}$ of closed ideals of [9]. In [11] one of the conditions require that the ideal I^* should also belong to the ideals defining the site, along with I. It seems to be an open problem, presumably difficult, to decide whether I^* is closed when I is. Thus, we don't know whether [11] applies to this topos.

3. It shows that Weil prolongations are sent into exponentials, a basic result that was missing in [11].

4. The definition of smooth topos is formulated directly in terms of C^∞-algebras, rather than ideals of definition of C^∞-algebras, These are less natural (for instance, several ideals may define the same algebra).

Theorem 2.5 *The embedding of the manifolds with boundary in a smooth topos*

$$\begin{cases} \mathcal{M}_\partial \longrightarrow \mathcal{E} \\ B \longmapsto s(B) = h_0^{C^\infty(B)} \end{cases}$$

is full and faithful, preserves open covers, products of the form $M \times B$ (M a manifold without boundary) and sends Weil prolongations into exponentials: $s({}^W B) = s(B)^{l(W)}$

Proof: Since Γ takes $s(f) : s(B) \longrightarrow s(C)$ into $f : B \longrightarrow C$, it is clear that the embedding is faithful. Let us prove that Weil prolongations are sent into exponentials, i.e. : $s({}^W B) = s(B)^{l(W)}$.

We have the following equivalences

$$\begin{array}{c|l} l(A) \longrightarrow s({}^W B) & \\ \hline C^\infty({}^W B) \longrightarrow A & \text{definition of } s({}^W B) \\ \hline C^\infty(B) \longrightarrow W \otimes A & \text{universal property of prolongations} \\ \hline l(W \otimes A) \longrightarrow s(B) & \text{Yoneda} \\ \hline l(W) \times l(A) \longrightarrow s(B) & \text{Yoneda sends coproducts into products} \\ \hline l(A) \longrightarrow s(B)^{l(W)} & \text{exponential adjointness} \end{array}$$

In particular, $\Gamma(s(B)^{l(W)}) = {}^W B$.

Let us turn to fullness. Assume that

$$s(B) \underset{g}{\overset{f}{\rightrightarrows}} s(C)$$

are such that $\Gamma(f) = \Gamma(g)$. We have to show that $f = g$. By embedding C as a closed set of some $\mathbb{R}^n$, it is enough to show that if $f : s(B) \longrightarrow R$ is 0 on points, then it is 0.

Assume that $\Gamma(f) = 0$. As before, it is enough to show that for all $\phi : l(W) \longrightarrow s(B)$, $f \circ \phi = 0$. Taking exponential adjoints, we have to show, equivalently, that the composite morphism in

$$1 \xrightarrow{\quad x \quad} s(B)^{l(W)} \xrightarrow{\quad f^{l(W)} \quad} R^{l(W)}$$

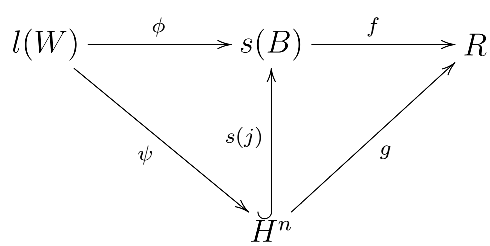
is 0.

But $x \in {}^W B$ and hence $x \in^W H_b$ for some $b \in B$, where $\{H_b\}_{b \in B}$ is a Euclidean cover of B. Therefore, either $\mathbb{H}^n \simeq H_b$ or $\mathbb{R}^n \simeq H_b$. We do only the first, since the second is similar, but simpler. Let $j : \mathbb{H}^n \hookrightarrow B$ be the obvious composition, $y \in \mathbb{H}^N$ such that $j(y) = x$ and lef $g = f \circ s(j) : H^n \longrightarrow R$. The global section of the composite of the two maps

$$R^n \xrightarrow{\quad sq_n \quad} H^n \xrightarrow{\quad g \quad} R$$

is 0. Indeed: $\Gamma(g \circ sq_n) = \Gamma(f) \circ j \circ \Gamma(sq_n) = 0$. This implies that $g \circ sq_n = 0$ which in turn implies that $g = 0$ by the density of sq_n.

Taking adjointness, we obtain the following commutative diagram



where ψ is the exponential adjoint of y. Therefore $f \circ \phi = g \circ \psi = 0$. This finishes the proof of fullness.

To show that open covers are preserved, we need a

Proposition 2.6 *The embedding $\mathcal{M}_\partial \longrightarrow \mathcal{E}$ preserves pull-backs of the form*

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{f} & C \\ \uparrow & & \uparrow \\ \mathbb{H}^n & \xrightarrow[g]{} & B \end{array}$$

where C is a manifold with boundary and $B \hookrightarrow C$ is a closed submanifold.

Proof: We prove, equivalently, that near-point determined C^∞-rings believe that the diagram

$$\begin{array}{ccc} C^\infty(C) & \xrightarrow{\rho'} & C^\infty(\mathbb{R}^n) \\ {\scriptstyle \tau'}\downarrow & & \downarrow{\scriptstyle \tau} \\ C^\infty(B) & \xrightarrow[\rho]{} & C^\infty(\mathbb{H}^n) \end{array}$$

is a push-out. Since near-point determined C^∞-rings can be embedded into direct products of Weil algebras, it is enough to consider Weil algebras. Let $\phi : C^\infty(\mathbb{R}^n) \longrightarrow W$ and $\psi : C^\infty(B) \longrightarrow W$ be such that $\phi \circ \rho' = \psi \circ \tau'$. Letting $\pi_0 : W \longrightarrow \mathbb{R}$ be the canonical map. $\pi_0 \circ \phi = ev(p)$ for a unique $p \in \mathbb{R}^n$. Similarly, $\pi_0 \circ \psi = ev(b)$ for a unique $b \in B$. But it is easy to check that $f(p) = b$ and this implies that $p \in \mathbb{H}^n$. By lemma 1.3, there is a unique $\theta : C^\infty(\mathbb{H}^n) \longrightarrow A$ such that $\theta \circ \tau = \phi$. We claim that $\theta \circ \rho = \psi$. But $\theta \circ \rho\tau' = \theta \circ \tau \circ \rho' = \phi \circ \rho' = \psi \circ \tau'$, and τ' is a quotient.

Corollary 2.7 *Assume that $j : H \hookrightarrow B$ is open in $\mathcal{M}_\partial$. Then the embedding $\mathcal{M}_\partial \longrightarrow \mathcal{E}$ preserves the pull-backs of the form*

$$\begin{array}{ccc} D(H) & \xhookrightarrow{D(j)} & D(B) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i'} \\ H & \xhookrightarrow{j} & B \end{array}$$

Proof: As already mentioned, it is enough to check this for Weil algebras.

Let $\phi : l(W) \longrightarrow s(D(H))$ and $\psi : l(W) \longrightarrow s(B)$ be such that $D(j) \circ \phi = i' \circ \psi$. Taking exponential adjoints and recalling that $s({}^W B) = s(B)^{l(W)}$ and $\Gamma(s(B)^{l(W)}) = {}^W B$, we are reduced to show that

$$\begin{array}{ccc} {}^W D(H) & \xhookrightarrow{D(j)} & {}^W D(B) \\ {\scriptstyle i} \uparrow & & \uparrow {\scriptstyle i'} \\ {}^W H & \xhookrightarrow{j} & {}^W B \end{array}$$

is a pull-back in sets. Take a Euclidean cover $\{H_b \hookrightarrow H\}_{b \in B}$ of H. Since $\bigcup_{b \in B} {}^W D(H_b) = {}^W D(H)$ and $\bigcup_{b \in B}^W H_b = {}^W H$ (cf. [2]) we are reduced to the preceding corollary.

Now we can finish the proof that the embedding $\mathcal{M}_\partial \longrightarrow \mathcal{E}$ preserves open covers: let $\{H_\alpha \hookrightarrow H\}_\alpha$ be an open cover of B. Then $D(H_\alpha) \hookrightarrow D(H)\}_\alpha$ is an open cover of $D(B)$. Since the embedding preserves open covers of manifolds without boundary (cf. [2])

$$\bigcup_\alpha s(D(H_\alpha) = s(D(H))$$

This implies that

$$\bigcup_\alpha s(D(H_\alpha)) \cap s(H) = s(H)$$

Invoking the last corollary, $s(D(H_\alpha)) \cap s(H) = s(H_\alpha)$.

For the preservations of products: Assume that M is a manifold without boundary, then

$$s(M \times B) = s(M) \times s(B)$$

Indeed, taking Euclidean covers for both M and B, say $M = \bigcup_{m \in M} U_m$ and $B = \bigcup_{b \in B} H_b$ and using preservation of covers, we are reduced to show (by distributivity of products over sums) that

$$\begin{cases} s(\mathbb{R}^n \times \mathbb{H}^m) = R^n \times H^m \\ s(\mathbb{R}^n \times \mathbb{R}^m) = R^n \times R^m \end{cases}$$

according to whether $H_b \simeq \mathbb{H}^m$ or $H_b \simeq \mathbb{R}^m$

But this is clear, since in the first case

$$\mathbb{R}^n \times \mathbb{H}^m = \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \mathbb{R}^{m-1} \simeq \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^{m-1} = \mathbb{H}^{m+n}$$

The same computation in the topos yields $R^n \times H^m \simeq H^{n+m}$. The second case is similar.

Proposition 2.8 *The embedding sends Weil prolongations into exponentials. More precisely,*

$$s({}^W B) = s(B)^{l(W)}$$

Proof: By using Yoneda lemma we can re-write 1.11 as the existence of natural (in A) bijections

$$\frac{h^A \longrightarrow h^{C^\infty({}^W\mathbb{R}^n)}}{h^A \longrightarrow (h^{C^\infty(\mathbb{R}^n)})^{h^W}} \qquad \frac{h^A \longrightarrow h^{C^\infty({}^W\mathbb{H}^n)}}{h^A \longrightarrow (h^{C^\infty(\mathbb{H}^n)})^{h^W}}$$

Recalling the definition of the embedding s and of l, this last assertion can be re-written, in turn, as

$$\begin{cases} s({}^W\mathbb{R}^n) = s(\mathbb{R}^n)^{l(W)} \\ s({}^W\mathbb{H}^n) = s(\mathbb{H}^n)^{l(W)} \end{cases}$$

Let $\{H_\alpha\}_\alpha$ be a Euclidean cover of B. By corollary 1.3 ${}^W B = \cup_\alpha^W H_\alpha$. Since open covers are preserved by s and $l(W)$ is an atom,

$$\begin{aligned} s({}^W B) &= \bigvee_\alpha s({}^W H_\alpha) \\ &= \bigvee_\alpha s(H_\alpha)^{l(W)} \\ &= (\bigvee_\alpha s(H_\alpha))^{l(W)} \\ &= s(B)^{l(W)} \end{aligned}$$

As examples of smooth topos we can mention the *Cahiers* topos in [2] and the topos of closed ideals in [9]. Thus this theorem applies to both. As far as I know, it is an open problem whether our main theorem can be extended to larger "smooth" toposes such as the topos $\mathcal{G}$ of germ-determined ideals (cf. e.g. [9]).

To finish, we should remark that this is not the first reworking of [11]. In fact a version of theorem 2.5 appears in Kock's book [4] III.9, page 252. As far as we can see however, he does not prove the preservation of open covers or the fact that Weil prolongations are sent into exponentials.

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