Lie derivatives, Lie brackets and vector fields over curves

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Although Lie derivatives can be defined for tensor fields quite generally, we shall limit ourselves to define this notion for vector fields and vector fields over curves.

0.1 Lie derivative of a vector field along a curve with respect to a vector field

Let M be a (microlinear) space, $Q : M \longrightarrow M^D$ a vector field, $\gamma : R \longrightarrow M$ an integral curve of Q and $W : R \longrightarrow M^D$ a vector field along γ. We define $L_Q W$ the *Lie derivative of* W *with respect to* Q to be the new vector field along γ characterized by the property

$$\begin{aligned} hL_Q W(t) &= Q^D_{-h}(W(t+h)) - W(t) \\ &= Q_{-h} \circ W(t+h) - W(t) \end{aligned}$$

for every $h \in D$. The fact that this equation characterizes $L_Q W$ is a consequence of Kock-Lawvere for the fiber $M_{\gamma(t)}$.

Proposition 0.1 *The following hold*

1. $L_Q(W_1 + W_2) = L_Q(W_1) + L_Q(W_2)$
2. $L_Q(fW) = f'W + fL_Q W$, *whenever* $f : R \longrightarrow R$.

Proof: This is a consequence of the following

Lemma 0.2 *The map $Q_d^D : M_{\gamma(t)} \longrightarrow M_{\gamma(t+d)}$ is R-linear.*

Proof: Indeed, $Q_d^D(v)(h) = Q(d, v(h))$. Hence,

$$\begin{array}{rcl} Q_d^D(\lambda v)(h) & = & Q(d, (\lambda v)(h)) \\ & = & Q(d, v(\lambda h)) \\ & = & Q_d^D(v)(\lambda h) \\ & = & \lambda Q_d^D(v)(h) \square \end{array}$$

The first is immediate by linearity of Q_d^D. The second is proved as in elementary calculus:

$$\begin{array}{rcl} hL_Q(fW)(t) & = & Q_{-h}^D(f(t+h)W(t+h)) - f(t)W(t) \\ & = & Q_{-h}^D(f(t+h)W(t+h)) - Q_{-h}^D(f(t)W(t+h)) \\ & + & Q_{-h}^D(f(t)W(t+h)) - f(t)W(t) \end{array}$$

Since $f(t+h) = f(t) + hf'(t)$, linearity of Q_h^D allows us to rewrite

$$\begin{array}{rcl} hL_Q(fW)(t) & = & hf'(t)Q_{-h}^D(W(t+h)) + f(t)[Q_{-h}^D(W(t+h)) - W(t)] \\ & = & hf'(t)Q_{-h}^D(W(t+h)) + hf(t)L_QW(t) \\ & = & hf'(t)W(t) + hf(t)L_QW(t)) \end{array}$$

To justify the last equality, it is enough to check that $hQ_{-h}^D(W(t+h)) = hW(t)$. But this is clear: indeed, define $\phi(h) = Q_{-h}^D(W(t+h)) \in M_{\gamma(t)}$ for all $h \in D$. Thus, $\phi \in M_{\gamma(t)}^D$. Since the Kock-Lawvere axiom holds for the fiber $M_{\gamma(t)}$, $\phi(h) = A + Bh$ for all $h \in D$, where $A, B \in M_{\gamma(t)}$. But $A = \phi(0) = W(t)$. Therefore $\phi(h) = W(t) + Bh$ and this implies that $h\phi(h) = hW(t)$, i.e. $hQ_{-h}^D(W(t+h)) = hW(t)$. This justifies the last equality and finishes the proof, by cancellation of the universally quantified h. An alternative proof proceeds by showing the (local) existence of a $\tilde{W}$ with $\tilde{W} \circ \gamma = W$ (cf. "Axiomsfieldequations(final).pdf", proposition 3.7) and then using the corresponding result for $[Q, \tilde{W}]$.

Define $W_d : R \longrightarrow M$ by the prescription $W_d(t) = W(t)(d)$ for $d \in D$.

Proposition 0.3 *The following are equivalent:*

1. *Every W_d is an integral curve of Q*

2. $L_QW = 0$

Proof: A simple computation:

$$\begin{array}{rll} (L_Q W)(t) = 0 & \text{iff} & \forall h \in D\ Q^D_{-h}(W(t+h)) = W(t) \\ & \text{iff} & \forall h \in D\ \forall d \in D\ Q(-h, W(t+h)(d)) = W(t)(d) \\ & \text{iff} & \forall h \in D\ \forall d \in D\ Q(-h, W_d(t+h)) = W_d(t) \end{array}$$

Since for all t $(L_Q W)(t) = 0$, we may substitute first h by $-h$ and then t by $t+h$ to obtain the equivalence between the RHS of last line and $Q(h, W_d(t)) = W_d(t+h)$ (for all $h, d \in D$ and all t,) i.e., for each d, W_d is an integral curve of Q.

Notice that W is completely determined by the family $\{W_d\}_d$ and vice-versa and both notions will be used interchangeably.

0.2 Lie derivative of a vector field relative to a vector field

Let $X, Y : M \longrightarrow M^D$ be vector fields. We define the *Lie derivative of* Y *with respect to* X by the formula

$$h(L_X Y)_x = (X_{-h})^D (Y_{X_h(x)}) - Y_x$$

(Notice the abuse of language here: $X_h(x) = X_x(h)$)

Proposition 0.4

1. $L_X(Y_1 + Y_2) = L_X(Y_1) + L_X(Y_2)$
2. $L_X(fY) = f L_X Y + X(f) Y$

Proof: Linearity of $(X_h)^D$. The first is immediate. As for the second,

$$\begin{array}{rcl} h(L_X fY)_x & = & (X_{-h})^D((fY)_{X_h(x)}) - (fY)_x \\ & = & (X_{-h})^D(f(X_h(x)) Y_{X_h(x)}) - f(x) Y_x \\ & = & (X_{-h})^D((f(x) + hX(f)(x)) Y_{X_h(x)} - f(x) Y_x \\ & = & f(x)(X_{-h})^D(Y_{X_h(x)}) + hX(f)(x) X^D_{-h}(Y_{X_h(x)}) - f(x) Y_x \\ & = & f(x)[(X_{-h})^D(Y_{X_h(x))}) - Y_x] + hX(f)(x) Y_{X_h(x)} \\ & = & hf(x)(L_X Y)_x + hX(f)(x) Y_x \end{array}$$

The last equality follows from $Y_{X_h(x)} = Y_x +$ terms in h

This Lie derivative is not new, since we knew it under another guise:

Proposition 0.5

$$L_XY = [X,Y]$$

Proof: (Lavendhomme page 237) By unravelling the definition

$$\begin{array}{rcl} h(L_XY)_x(d) & = & [(X_{-h})\circ(Y_{X_h(x)}) - Y_x](d) \\ & = & X_{-h}\circ Y_d\circ X_h(x) - Y_d(x) \end{array}$$

Define $l_h(d,d') = (X_{-h}\circ Y_d\circ X_h\circ Y_{-d'})_x$ Then

$$\begin{cases} l_h(d,0) = (X_{-h}\circ Y_d\circ X_h)_x \\ l_h(0,d) = (Y_{-d})_x \end{cases}$$

Thus, recalling the definition of addition on fibers,

$$\begin{array}{rcl} h(L_XY)_x(d) & = & l_h(d,d) \\ & = & (X_{-h}\circ Y_d\circ X_h\circ Y_{-d})_x \\ & = & [Y,X]_x(h(-d)) \\ & = & [X,Y]_x(hd) \\ & = & h[X,Y]_x(d) \end{array}$$

(For the properties of $[-.-]$ see "curvaturetorsion.pdf", page 9)

The Lie bracket and the Lie derivative are connected as follows:

Proposition 0.6 *If $Q\in\mathcal{X}(M)$, $\gamma : R\longrightarrow M$ an integral curve of Q, $W : R\longrightarrow M^D$ a vector field along γ and $\tilde{W}\in\mathcal{X}(M)$ is such that $\tilde{W}\circ\gamma = W$, then*

$$L_QW = [Q,\tilde{W}]\circ\gamma$$

Proof: This is practically a repetition of the proof that $L_XY = [X,Y]$. Recall that L_QW is (uniquely) characterized by the equation

$$hL_QW(t) = Q^D_{-h}(W(t+h)) - W(t)$$

for all $h\in D$. A simple computation allows us to write

$$\begin{array}{rcl} h(L_QW)(t))_d & = & (Q_h\circ W(t+h) - W(t))_d \\ & = & (Q_{-h}\circ\tilde{W}_d\circ Q_h)(\gamma(t)) - \tilde{W}_d(\gamma(t) \end{array}$$

In fact, by hypothesis, $W(t) = \tilde{W}\circ\gamma$ and $Q_h(\gamma(t)) = \gamma(t+h)$, since γ is an integral curve of Q.

Define $l_h(d,d') = (Q_{-h} \circ \tilde{W}_d Q_h \circ \tilde{W}_{-d'})(\gamma(t))$. Then

$$\begin{cases} l_h(d,0) = (Q_h \circ \tilde{W}_d \circ)(\gamma(t)) \\ l_h(0,d) = \tilde{W}_{-d'}(\gamma(t)) = -\tilde{W}_{d'}(\gamma(t)) \end{cases}$$

Therefore, by definition of the addition on fibers

$$\begin{array}{rcl} h(L_Q W)(t)(d) & = & (Q_{-h} \circ \tilde{W}_d \circ Q_h)(\gamma(t)) - \tilde{W}_d(\gamma(t) \\ & = & l_h(d,d) \\ & = & (Q_h \circ \tilde{W}_d \circ Q_h \circ \tilde{W}_{-d})(\gamma(t)) \\ & = & [\tilde{W}, Q]_{\gamma(t)}(h.(-d)) \\ & = & [Q, \tilde{W}]_{\gamma(t)}(hd) \\ & = & h([Q, \tilde{W}] \circ \gamma)(t)(d) \end{array}$$

Corollary 0.7 *Let $\gamma : R \longrightarrow M$ be an integral curve of Q. Assume, furthermore, that $\tilde{W} \in \mathcal{X}(\mathcal{M})$ is such that $W = \tilde{W} \circ \gamma$. Then the following are equivalent*

1. *Every γ_h is an integral curve of Q*
2. $L_Q W = 0$
3. $[Q, \tilde{W}] \circ \gamma = 0$

0.3 Computations for $M = R^n$

Let $W : R \longrightarrow M^D$ be a vector field over $\gamma : R \longrightarrow M$ and $M = R^n$. Then $M^D = M \times R^n$ and W is completely determined by n functions $f^i : R \longrightarrow R$ by the equation (Kock-Lawvere)

$$W(t) = [d \longrightarrow \gamma(t) + d(f^1(t), \ldots, f^n(t))]$$

Alternatively, this equation may be re-written as

$$W(t) = \sum_{i=1}^{n} f^i(t)(\partial/\partial x^i)_{\gamma(t)}$$

with $(\partial/\partial x^i)_x = [d \mapsto x + d(0, \ldots, 1, \ldots, 0)]$ (the 1 in the i^{th} place). Notice that the sequence $\{(\partial/\partial_1)_x. \ldots (\partial/\partial_n)_x\}$ form a basis of M_x.

Proposition 0.8 *If*

$$\begin{cases} X = \sum_{i=1}^n a^i \partial/\partial x^i \\ Y = \sum_{i=1}^n b^j \partial/\partial x^i \end{cases}$$

then

$$L_X Y = \sum_{i=1}^n \sum_{j=1}^n [a^i(\partial b^j/\partial x^i)\partial/\partial x^j - b^j(\partial a^i/\partial x^j)\partial/\partial x^i]$$

Proof: Straightforward computation from the properties of L_X

$$\begin{array}{rcl} (*) \quad L_X Y & = & \sum_{i=1}^n L_X(b^i \partial/\partial x^i) \\ & = & \sum_{i=1}^n [X(b^i)\partial/\partial x^i + b^i L_X \partial/\partial x^i] \end{array}$$

On the other hand,

$$\begin{array}{rcl} L_X \partial/\partial x^i & = & -[\partial/\partial x^i, X] \\ & = & -\sum_{j=1}^n L_{\partial/\partial x^i}(a^j \partial/\partial x^j) \\ & = & -\sum_{j=1}^n (\partial a^j/\partial x^i)\partial/\partial x^j + a^j[\partial/\partial x^i, \partial/\partial x^j] \\ & = & -\sum_{j=1}^n (\partial a^j/\partial x^i)\partial/\partial x^j \end{array}$$

since $[\partial/\partial x^i, \partial/\partial x^j] = 0$.

Finally, $X(b^i) = \sum_{j=1}^n a^j \partial/\partial x^j (b^i) = \sum_{j=1}^n a^j \partial b^i/\partial x^j$

Substituting in $(*)$, we obtain

$$\begin{array}{l} L_X Y = \sum_{i=1}^n [\sum_{j=1}^n a^j(\partial b^i/\partial x^j)\partial/\partial x^i - b^i \sum_{j=1}^n (\partial a^j/\partial x^i)\partial/\partial x^j] \\ = \sum_{i=1}^n \sum_{j=1}^n a^j(\partial b^i/\partial x^j)\partial/\partial x^i - \sum_{i=1}^n b^i \sum_{j=1}^n (\partial a^j/\partial x^i)\partial/\partial x^j \\ = \sum_{i=1}^n \sum_{j=1}^n a^j(\partial b^i/\partial x^j)\partial/\partial x^i - \sum_{i=1}^n \sum_{j=1}^n b^i(\partial a^j/\partial x^i)\partial/\partial x^j \\ = \sum_{i=1}^n \sum_{j=1}^n [a^j(\partial b^i/\partial x^j)\partial/\partial x^i - b^i(\partial a^j/\partial x^i)\partial/\partial x^j] \end{array}$$

By interchanging i and j we obtain the desired result:

$$L_X Y = \sum_{i=1}^n \sum_{j=1}^n [a^i(\partial b^j/\partial x^i)\partial/\partial x^j - b^j(\partial a^i/\partial x^j)\partial/\partial x^i]$$

By further interchanging i and j in the second double sum we obtain

$$\begin{array}{rcl} L_X Y & = & \sum_{i=1}^n \sum_{j=1}^n [a^i(\partial b^j/\partial x^i)\partial/\partial x^j - b^i(\partial a^j/\partial x^i)\partial/\partial x^j] \\ & = & \sum_{j=1}^n [\sum_{i=1}^n a^i(\partial b^j/\partial x^i) - b^i(\partial a^j/\partial x^i)]\partial/\partial x^j \end{array}$$

which is the formula given by Spivak I, page 212.

In particular, the j^{th}-component of $L_X Y$ is

$$(L_X Y)^j = \sum_{i=1}^n [a^i(\partial b^j/\partial x^i) - b^i(\partial a^j/\partial x^i)]$$