General Relativity: Metrics, connections and curvature

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This pamphlet fits squarely in (classical) differential geometry: metrics and its relation to connections and curvature.

0.1 Metrics on a manifold

Let M be a n-dimensional manifold. A metric g on M is a map $g : M^D \times_M M^D \longrightarrow R$ satisfying the following requirements for every $x \in M$:

(1) $g_x : M_x \times M_x \longrightarrow R$ is bilinear

(2) $g_x(u, v) = g_x(v, u)$ whenever $u, v \in M_x$ (symmetry)

(3) If $g_x(u, v) = 0$ for all $v \in M_x$, then $u = 0$ (non-degeneracy)

Notice that for an R-module V we have an equivalence

$$\frac{\phi : V \times V \longrightarrow R \quad \text{bilinear}}{\psi : V \longrightarrow V^* \quad \text{linear}} \qquad \psi(u)(v) = \phi(u, v)$$

where $V^* = Lin_R(V, R)$. In particular (for $V = M_x$ and $\phi = g_x$) from (1) we obtain a linear map

$$\psi : M_x \longrightarrow (M_x)^*$$

The map ψ is injective (because of (3)) and hence it preserves linear independence. Furthermore ψ is surjective, since M_x and $(M_x)^*$ have the same dimension. In fact,

Lemma 0.1 *If an R-module V is n-dimensional, then so is V^*.*

Proof: The usual proof for vector spaces (see e.g [?]) apply without changes to this case.

Let $\{\partial/\partial x^i|_x, \ldots, \partial/\partial|_x\}$ be the canonical basis of $M_x = R^n$ and $\{\epsilon^1, \ldots, \epsilon^n\}$ the dual basis of $(M_x)^*$. Then,

$$\psi(\partial/\partial x^j|_x) = \sum_i \lambda_i^j \epsilon^i$$

Relative to these bases, the matrix of ψ is $(\lambda_j^i)_{ij}$. To compute the coefficients of this matrix, notice that from the previous expression, $\psi(\partial/\partial x^j|_x)(\partial/\partial x^k|_x) = \sum_i \lambda_i^j \epsilon^i(\partial/\partial x^k|_x) = \sum_i \lambda_i^j \delta_k^i = \lambda_k^j$, i.e., $\lambda_i^j = g_{ji}$. Thus, the matrix sought for is $M(\psi) = (g_{ij})_{ij}$.

Since ψ has an inverse, let $M(\psi^{-1}) = (g^{ij})_{ij}$ be the matrix of this inverse relative to the same bases. Then $(g_{ij})_{ij}$ and $(g^{ij})_{ij}$ are inverse of each other. They will play a fundamental role in "axiomsfieldequations(final).pdf".

0.2 Metrics and associated connections

Given a metric g we shall define a connection $\nabla : M^D \times_M M^D \longrightarrow M^{D\times D}$ as follows. Recall that by introducing coordinates we may as well assume that $M = R^n$. In this context, a connection $\nabla : R^n \times R^n \times R^n \longrightarrow R^n \times R^n \times R^n \times R^n \longrightarrow R^n$ is completely determined by its fourth component $\nabla^4 : R^n \times R^n \times R^n \longrightarrow R^n$. Let $\Gamma_{ij}^k(x) = -(\nabla^4(\partial/\partial x^i|_x, \partial/\partial x^j|_x))_k$, where $(v)_k$ denotes the k^{th}-component of v. We define

$$g_{ij}(x) = g(\partial/\partial x^i|_x, \partial/\partial x^j|_x)$$

From now on, we shall write $< -, - >$ for a metric, rather than $g(-,-)$. We say that a connection ∇ *is compatible with a metric* $< -, - >$ if the parallel transport along any curve γ, $\tau_h(\gamma, -) : \pi_M^{-1}(\gamma(t)) \longrightarrow \pi_M^{-1}(\gamma(t+h))$ is an isometry for the metric $< -, - >$.

Proposition 0.2 *A connection ∇ is compatible with a metric $< -, - >$ iff for every curve γ and every $t \in R$, $\tau_t(\gamma) : M_{\gamma(0)} \longrightarrow M_{\gamma(t)}$ is an isometry for $< -, - >$.*

Proof: By definition, $\tau_t(\gamma, v) = V(t)$. Now,

$$\begin{aligned} hd/dt < V(t), V(t) > &= < V(t+h), V(t+h) > - < V(t), V(t) > \\ &= < \tau_h V(t), \tau_h V(t) > - < V(t), V(t) > \end{aligned}$$

By hypothesis, $< \tau_h V(t), \tau_h V(t) >=< V(t), V(t) >$. Thus,

$$d/dt < V(t), V(t) >= 0$$

and hence $< V(t), V(t) >$ is constant.

Lemma 0.3 *A connection ∇ is compatible with a metric $< -, - >$ iff it satisfies the following condition: whenever V, W are vector fields along any curve γ,*

$$d/dt < V, W >=< DV/dt, W > + < V, DW/dt >$$

Proof: [cf. Spivak, volume II, page 254] Assume that ∇ satisfies the condition and let $v \in M_{\gamma(0)}$. Let V be the unique parallel vector field along γ such that $V(0) = v$. Then

$$d/dt < V, V >= 2 < DV/dt, V >= 0$$

So $< V, V >$ is constant along γ and in fact equal to $< v.v >$.

We recall that, by definition, $\tau_t(\gamma, v) = V(t)$. Thus, $\tau_t(\gamma, -)$ is an isometry, i.e., ∇ is compatible with the metric.

Assume now that ∇ is compatible with the metric $< -, - >$. Choose parallel vector fields $P_1, \ldots P_n$ along a curve γ, which are orthonormal at one point of γ and hence at every point of γ. Let

$$\begin{cases} V(t) = \sum_{i=1}^n v^i(t) P_i(t) \\ W(t) = \sum_{j=1}^n w^j(t) P_j(t) \end{cases}$$

Therefore

$$< V(t), W(t) >= \sum_{i=1}^n v^i(t) w^i(t)$$

i.e.,

$$< V, W >= \sum_{i=1}^n v^i w^i$$

By the properties of covariant derivation and the fact that the P's are parallel along γ,

$$\begin{cases} DV/dt = \sum_{i=1}^n Dv^i/dtP_{i_t} \\ DW/dt = \sum_{j=1}^n Dw^i/dtP_{j_t} \end{cases}$$

Hence,

$$\begin{aligned} < DV/dt, W > + < V, DW/dt > &= \textstyle\sum_{i+1}^n [(Dv^i/dt)w^i + v^i Dw^i/dt] > \\ &= d/dt < V, W > \end{aligned}$$

Corollary 0.4 *The connection ∇ is compatible with the metric $< -, - >$ iff*

$$\partial g_{jk}/\partial x^i =< \nabla_{\partial/\partial x^i}\partial/\partial x^j, \partial/\partial x^k > + < \partial/\partial x^j, \nabla_{\partial/\partial x^i}\partial/\partial x^k >$$

Proof: Assume that ∇ is compatible with $< -, - >$ and let $x_0 = (x_0^1, \ldots, x_0^n) \in M = R^n$. Define $\gamma(t) = (x^1(t), \ldots, x^n(t))$ with $x^i(t) = x_0^i + t$ and $x^j(t) = x_0^j$ for all $j \neq i$. By the previous proposition,

$$\begin{aligned} d/dt < \partial/\partial x^j \circ \gamma, \partial/\partial x^k \circ \gamma > &= < D/dt(\partial/\partial x^j \circ \gamma), \partial/\partial x^k \circ \gamma > \\ &+ < \partial/\partial x^j \circ \gamma, D/dt(\partial/\partial x^k \circ \gamma) > \end{aligned}$$

But

$$\begin{cases} d/dt = \partial/\partial x^i \\ D/dt(\partial/\partial x^j \circ \gamma) = \nabla_{\partial/\partial x^i}\partial/\partial x^j \end{cases}$$

In fact, the first is obvious and the second follows from "covariant derivation.pdf" page 5 (Proposition 6 (c)). Computing at $t = 0$, concludes the proof in one direction. To prove the other direction, let $\gamma(t) = (x^1(t), \ldots, x^n(t))$ be a curve of $M = R^n$. The proof proceeds by writing

$$\begin{cases} V(t) = \sum_i v^i(t)\partial/\partial x^i|_{\gamma(t)} \\ W(t) = \sum_i w^i(t)\partial/\partial x^i|_{\gamma(t)} \end{cases}$$

Bi-linearity of the metric reduces the computation of $d/dt < V(t), W(t) >$ to the special case: $V(t) = \partial/\partial x^j|_{\gamma(t)}$ and $W(t) = \partial/\partial x^k|_{\gamma(t)}$.

Lemma 0.5

$$< D/dt(\partial/\partial x^j|_{\gamma(t)}, \partial/\partial x^k|_{\gamma(t)} >= \sum_i < \nabla_{\partial/\partial x^i}\partial/\partial x^j|_{\gamma(t)}, \partial/\partial^k|_{\gamma(t)} > dx^i/dt$$

Proof: This follows from "covariantderivation.pdf" pages 5:

$$D/dt(\partial/\partial x^j|_{\gamma(t)}) = \sum_{i,l} \Gamma^l_{ij} \partial/\partial x^l|_{\gamma(t)} dx^i/dt$$
$$\nabla_{\partial/\partial x^i} \partial/\partial x^j|_{\gamma(t)} = \sum_{i,l} \Gamma^l_{ij} \partial/\partial x^l|_{\gamma(t)}$$

To finish the proof that ∇ is compatible with the metric, we just compute

$$\begin{aligned}
& d/dt < \partial/\partial x^j|_{\gamma(t)}, \partial/\partial x^k|_{\gamma(t)} >= \sum_i \partial/\partial x^i < \partial/\partial x^j|_{\gamma(t)}, \partial/\partial x^k|_{\gamma(t)} > dx^i/dt \\
& = \sum_i < \nabla_{\partial/\partial x^i|_{\gamma(t)}} \partial/\partial x^j, \partial/\partial x^k|_{\gamma(t)} > dx^i/dt \\
& + \sum_i < \partial/\partial x^i|_{\gamma(t)}, \nabla_{\partial/\partial x^i} \partial/\partial x^k > dx^i/dt \\
& =< D/dt \partial/\partial x^j|_{\gamma(t)}, \partial/\partial x^k > + < \partial/\partial x^j, D/dt \partial/\partial x^k|_{\gamma(t)} >
\end{aligned}$$

Proposition 0.6 (Fundamental lemma of Riemannian Geometry) *On a Riemannian manifold there is a unique symmetric connection compatible with the metric.*

Proof: Define

$$\begin{cases} [ij, l] = 1/2[\partial g_{il}/\partial x^j + \partial g_{jl}/\partial x^i - \partial g_{ij}/\partial x^l] \\ \Gamma^k_{ij} = \sum_{l=1}^n g^{kl}[ij, l] \end{cases}$$

As we saw in ??, the Γ^k_{ij}'s define ∇ completely. It is easy to see that the connection thus defined is symmetric.

Notice that the second equation is equivalent to $[ij, l] = \sum_{k=1}^n \Gamma^k_{ij} g_{lk}$, since the matrices $(g^{kl})_{kl}$ and $(g_{lk})_{lk}$ are inverse of each other.

Claim 1: $\nabla_{\partial/\partial x^i} \partial/\partial x^j = \sum_{k=1}^n \Gamma^k_{ij} \partial/\partial x^k$ (cf. covariantderivation.pdf page 6)

Claim 2: $< \nabla_{\partial/\partial x^i} \partial/\partial x^j, \partial/\partial x^l >= [ij, l]$. In fact,

$$\begin{aligned}
< \nabla_{\partial/\partial x^i} \partial/\partial x^j, \partial/\partial x^l > &= < \sum_{k=1}^n \Gamma^k_{ij} \partial/\partial x^k, \partial/\partial x^l > \\
&= \sum_{k=1}^n \Gamma^k_{ij} < \partial/\partial x^k, \partial/\partial x^l > \\
&= \sum_{k=1}^n \Gamma^k_{ij} g_{kl} \\
&= [ij, l]
\end{aligned}$$

Claim 3: $\partial g_{jk}/\partial x^i = [ij, k] + [ik, j]$. Indeed, this is obvious from the very definition of $[ij, l]$.

Notice that we may re-write claim 3 as

$$\partial/\partial x^i g_{jk} =< \nabla_{\partial/\partial x^i} \partial/\partial x^j, \partial/\partial x^k > + < \partial/\partial x^j, \nabla_{\partial/\partial x^i} \partial/\partial x^k >$$

Corollary 0.7 *The following are equivalent:*

1. $\Gamma^k_{ij}(x) = 0$ *for all* i, j, k
2. $(\partial g_{ij}/\partial x^k)(x) = 0$ *for all* i, j, k

Proof: Assume *1*. Then $[ij, l] = \sum_{k=1}^n \Gamma^k_{ij} g_{lk} = 0$ (proof of claim 2) and hence *2* by claim 3. Assume *2*. Then $[ij, l] = 0$ (by definition of $[ij, l]$) and this implies *1* by the very definition of Γ^k_{ij}.

Claim 4: ∇ is compatible with $< -, - >$. This was already proved (cf. 0.4)

Claim 5: The connection defined above is the only symmetric connection compatible with the given metric. (Cf [**?**] II, page 256.)

To show this, let ∇ be any symmetric connection compatible with the metric, i.e., claim 3 holds. By defining $[ij, k] =< \nabla_{\partial/\partial x^i} \partial/\partial x^j, \partial/\partial x^k >$, we obtain by cyclically permuting i, j, k

$$\begin{aligned} \partial g_{jk}/\partial x^i &= [ij, k] + [ik, j] \\ \partial g_{ki}/\partial x^j &= [jk, i] + [ji, k] \\ \partial g_{ij}/\partial x^k &= [ki, j] + [kj, i] \end{aligned}$$

Using symmetry, i.e., $[ij, k] = [ji, k]$, a trivial computation gives

$$1/2[\partial g_{ik}/\partial x^j + \partial g_{jk}/\partial x^i - \partial g_{ij}/\partial x^k] = [ij, k]$$

This is the defining formula for $[ij, k]$ used above to obtain a connection from a metric. We had already proved that

$$(\nabla_{\partial/\partial x^i} \partial/\partial x^j)_x = \sum_{k=1}^n \Gamma^k_{ij}(x) \partial/\partial x^k|_x$$

Therefore,

$$\begin{aligned} [ij, k] &= < \nabla_{\partial/\partial x^i} \partial/\partial x^j, \partial/\partial x^k > \quad \text{(by definition)} \\ &= < \textstyle\sum_{l=1}^n \Gamma^l_{ij} \partial/\partial x^l, \partial/\partial x^k > \\ &= \textstyle\sum_{l=1}^n \Gamma^l_{ij} < \partial/\partial x^l, \partial/\partial x^k > \\ &= \textstyle\sum_{l=1}^n \Gamma^l_{ij} g_{lk} \end{aligned}$$

But this implies that

$$\Gamma^l_{ij} = \sum_{k=1}^n g^{kl}[ij, k],$$

i.e., the same formula to define the connection from a metric used above.

The unique symmetric connection compatible with a metric is called, improperly according to Spivak ([?] II, page 256), the *Levi-Civita connection* for the metric.

0.3 Curvature in terms of the metric

Assume that we are given a microlinear space M with a metric g. As shown previously ??? there is exactly one connection compatible with g, namely the one defined by

$$\begin{cases} [ij,l] = 1/2[\partial g_{il}/\partial x^j + \partial g_{jl}/\partial x^i - \partial g_{ij}/\partial x^l] \\ \Gamma^k_{ij} = \sum_{l=1}^n g^{kl}[ij,l] \end{cases}$$

Claim 1: $[ij,l] = \sum_{k=1}^n g_{lk}\Gamma^k_{ij}$

This is obvious from the definition of $(g_{lk})_{lk}$ as the inverse matrix of $(g^{kl})_{kl}$.

Claim 2: $\partial g_{jk}/\partial x^i = [ij,k] + [ik,j]$

This was proved in "axiomsfieldequations(final).pdf" (pages 13/14)

Claim 3:

$$\sum_{\gamma=1}^n g_{i\gamma}\partial\Gamma^\gamma_{jk}/\partial y^l \quad = \quad \partial[jk,i]/\partial y^l - \sum_{\gamma=1}^n \Gamma^\gamma_{jk}([il,j] + [\gamma l,i])$$

This follows from the product rule for differentiation and claims 1,2

$$\sum_{\gamma=1}^n g_{i\gamma}\partial\Gamma^\gamma_{jk}/\partial y^l \quad = \quad \partial/\partial y^l(\sum_{\gamma=1}^n g_{i\gamma}\Gamma^\gamma_{jk}) \quad - \quad \sum_{\gamma=1}^n(\partial g_{i\gamma}/\partial y^l)\Gamma^\gamma_{jk}$$

For the next claim, define

$$R_{iljk} = \sum_\gamma g_{ij}R^\gamma_{jlk}$$

Claim 4

$$R_{ijlk} = 1/2[\partial^2 g_{ik}/\partial y^j\partial y^l + \partial^2 g_{jl}/\partial y^i\partial y^k - \partial^2 g_{il}/\partial y^j\partial y^k - \partial^2 g_{jk}/\partial y^i\partial y^l]$$

This follows from our expression for the Riemann-Christoffel tensor (see "Riemann-Christoffel.pdf" page 3):

$$R^\gamma_{jlk} = \partial\Gamma^\gamma_{kj}/\partial y^l - \partial\Gamma^\gamma_{lj}/\partial y^k + \sum_{\mu=1}^n(\Gamma^\mu_{kj}\Gamma^\gamma_{l\mu} - \Gamma^\mu_{lj}\Gamma^\gamma_{k\mu})$$

and our previous claims. In more detail:

$$\begin{array}{rcl} R_{iljk} & = & \sum_\gamma g_{ij} R^\gamma_{jlk} \\ & = & \sum_\gamma g_{ij} \partial \Gamma^\gamma_{kj}/\partial y^l - \sum_\gamma g_{ij} \partial \Gamma^\gamma_{lj}/\partial y^k \\ & + & \sum_\gamma g_{ij} \sum_\mu \Gamma^\mu_{kj}\Gamma^\gamma_{l\mu} - \sum_\gamma g_{ij} \sum_\mu \Gamma^\mu_{lj}\Gamma^\gamma_{k\mu} \end{array}$$

Now use claims 1,2,3.

Claim 5

$$R_{ijkl}(p) =< R_{\partial/\partial y^l|_p, \partial/\partial y^k|_p} \partial/\partial y^j|_p, \partial/\partial y^i|_p >$$

In terms of the canonical basis $\{\partial/\partial y^1, \ldots, \partial/\partial y^n\}$

$$\begin{array}{rcl} RHS & = & < \sum_\gamma (R_{\partial/\partial y^k|_p, \partial/\partial y^l|_p} \partial/\partial y^j|_p) \partial/\partial y^\gamma, \partial/\ \partial y^i > \\ & = & \sum_\gamma R_{\partial/\partial y^k|_p, \partial/\partial y^l|_p} \partial/\partial y^j < \partial y^\gamma, \partial y^i > \\ & = & \sum_\gamma R_{\partial/\partial y^k|_p, \partial/\partial y^l|_p} \partial/\partial y^j g_{\gamma i} \\ & = & \sum_\gamma R^\gamma_{jkl} g_{i\gamma} \\ & = & R_{ijkl} \end{array}$$

Proposition 0.8 *The following holds*

$$\begin{array}{lllll} (1) & < R_{XY}Z, W > & = & - < R_{YX}Z, W > \\ (2) & < R_{XY}Z, W > & = & - < R_{XY}W, Z > \\ (3) & \sum_0 < R_{XY}Z, W >= 0 & & \\ (4) & < R_{XY}Z, W > & = & < R_{ZW}X, Y > \end{array}$$

(In (3) the cyclic permutations apply only to X, Y, Z.)

Proof: (1) follows from $R_{XY}Z = -R_{YX}Z$ and (3) from the Jacobi identity. To prove (2) and (4), introduce a coordinate system. From Claim 5, (2) and (4) become, respectively

$$\begin{array}{ll} (2)' & R_{ijkl} = -R_{jikl} \\ (4)' & R_{klij} = R_{ijkl} \end{array}$$

But these follow at once from Claim 4.

Corollary 0.9 *Ric is symmetric, i.e.,* $Ric(u, v) = Ric(v, u)$.

Proof: Recall that

$$Ric(u,v) = \sum_i < R_{u,\partial/\partial x^i} v, \partial/\partial x^i >$$

and that we can express the difference (cf. "Riemann-Christoffel.pdf" page 5)

$$Ric(u,v) - Ric(v,u) = \sum_i < R_{uv} \partial/\partial x^i, \partial/\partial x^i >$$

But each term of the sum is zero by (2).