A model of SDG in which only trivial distributions with compact support have a density

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Abstract

We exhibit a model of SDG having the property that only trivial distributions of compact support on R^m have a density.

The aim of this note is to show that in the C^ω-ring classifier the only distributions with compact support on R^m having a density are trivial in a sense to be specified. This was conjectured by Anders Kock in [2]. I would like to thank my colleague Paul M. Gauthier for suggesting a proof of the characterization of functions with compact support of the Appendix. My thanks go also to Bill Lawvere, who read a first draft, pointed out some omissions and made some suggestions that resulted in several improvements in the text.

1 The model

Recall that the C^ω*-ring classifier* is the topos defined as

$$\mathcal{C}^\omega = Sets^{C^\omega}$$

where C^ω is the category of finitely presented C^ω-algebras of the form $C^\omega(\mathbb{R}^k)/I$ (I finitely generated).

Let us recall the *dictionary* (see [4] for the smooth case) which expresses the basic notions of the model such as "reals" (elements of

R), "ring structure of R", "infinitesimals", "positive reals", etc. at a stage $A = C^\omega(\mathbb{R}^k)/I$ in terms of the notions of C^ω-functions and their ideals:

> A *real* at stage A is an equivalence class $f(x)$ mod I, where $f \in C^\omega(\mathbb{R}^k)$, i.e., an element of A. We shall write $[f(x)]_I$ for this equivalence class. Thus, a representative is a real $f(x) \in \mathbb{R}$ depending analytically on the parameter x.
>
> The *ring structure* of R at stage A is given by
>
> $$0 = |0|_I, 1 = |1|_I$$
> $$|f|_I + |g|_I = |f + g|_I$$
> $$|f|_I.|g|_I = |f.g|_I$$
>
> A *positive real* at stage A is an equivalence class $|f(x)|_I$ such that there is an open subset set $U \supset Z(I)$ with the property that for every $x \in U$, $f(x) > 0$.
>
> A *natural number* at stage A is an equivalence class of a constant function whose value is a natural number.
>
> A *real-valued function* on m variables at stage A is an equivalence class $|f(s,x)|_{I^*}$ where $f \in C^\omega(\mathbb{R}^m \times \mathbb{R}^k)$. Thus, a representative is a function $f(-,x)$ depending analytically on the parameter x.

We assume that the reader is familiar with the clauses defining the forcing relation $\Vdash$ interpreting the logical connectives in a topos. (See e.g. [4])

Lemma 1.0.1 *Let $A = C^\omega(\mathbb{R}^n)/I$ be a finitely generated C^ω-algebra and let $a = [f(x)]_I$. Then the following conditions are equivalent:*

(1) $A \Vdash \forall \epsilon > 0 \; |a| < \epsilon$

(2) $f|_{Z(I)} \equiv 0$

Proof:
(1)→(2): Assume *(1)*. Then, for every positive real number ϵ there is an open $U \supset Z(I)$ such that for every $x \in U$ $|f(x)| < \epsilon$. In particular this holds for every $x \in Z(I)$, proving *(2)*.

(2)→*(1)*: Assume *(2)*. Let $H : A \longrightarrow B$ be a morphism of C^ω-algebras with $B = C^\omega(\mathbb{R}^m)/J$ and let $E(y) \in C^\omega(\mathbb{R}^m)$ be positive at stage B in the sense that $E(y)|_{Z(J)} > 0$. Let $h : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a representative of H. Recall that h takes zeros of J into zeros of I and this implies that

$$f(h(y))|_{Z(J)} \equiv 0$$

Therefore, for some $U \supset Z(J)$ $|f(h(y))| < E(y)$, by continuity of the functions involved.

Remark 1.0.2 It is clear that $\mathbb{C}^\omega \Vdash \forall a \in R \ (\exists k \ (a^k = 0) \rightarrow \forall \epsilon > 0 \ |a| < \epsilon)$. On the other hand, the converse does not hold. In fact, the Weierstrass' factor theorem (see e.g [1] or [3]) implies the existence of an analytic function $f(x)$ with the same zeros as $sin\ x$, but whose orders are unbounded. Let $A = C^\omega(\mathbb{R})/I$ with $I = (f)$. If we let $a = [f(x)]_I$, then the lemma tells us that a satisfies *(1)* (of the lemma). On the other hand, for every k, $sin^k(x) \notin I$. In fact, assume that $sin^k(x) = g(x)f(x)$ for some k. Taking the k^{th} derivative on both sides, we obtain an expression of the form $k!\ cos\ x + sin\ x\ (-) = \sum_{i=0}^{k} g^{(i)}(x) f^{(k-i)}(x)$. Pick a zero x_0 of f of order $> k$. Computing both members of this equation at x_0. we obtain $k! = 0$, a contradiction.

Proposition 1.0.3 *Let $A = C^\omega(\mathbb{R}^m)/I$ be a finitely generated C^ω-algebra and let $\phi = [f(s,x)]_{I^*}$ be a function on l variables defined at stage A. Then the following are equivalent*

(1) $A \Vdash \forall \psi \ \exists b > 0 \ \forall n \in N | \int_{R_n} \phi\psi | < b$

(2) $f|_{\mathbb{R}^l \times Z(I)} \equiv 0$

(3) $A \Vdash \forall \psi \ \forall b > 0 \ \forall n \in N | \int_{R_n} \phi\psi | < b$

where $\int_{R_n}$ is the distribution (of compact support) defined by the iterated integral $\int_{R_n} \phi = \int_{-n}^{n} \dots \int_{-n}^{n} \phi(x_1, \dots, x_l) dx_1 \dots dx_l$.

Proof:

(1)→*(2)*: Assume *(1)*. Then for every $g(s)$ there is a function $B(x)$ such that for every $n \in \mathbb{N}$ there is an open $U_n \supset Z(I)$ satisfying

$$|\int_{R_n} f(s,x)g(s)ds| < B(x)$$

for every $x \in U_n$. Hence, if $x_0 \in Z(I)$, the sequence $\{|\int_{R_n} f(s,x_0)g(s)ds|\}_n$ is bounded. By the main lemma from analysis (cf. Appendix) it follows that $f(-,x_0) \equiv 0$, i.e. we have proved *(2)* since x_0 was an arbitrary element in $Z(I)$.

(2)$\rightarrow$*(3)*: We proceed as in the proof of *(2)*$\rightarrow$*(1)* of the lemma, keeping the notations thereof. Let $\psi = g(s,y)$ and let $n \in \mathbb{N}$. From *(2)* it follows that $f(s,h(y))|_{Z(J^*)} \equiv 0$. In turn, this implies that $\int_{R_n} f(s,h(y))g(s,y)ds|_{Z(J)} \equiv 0$. The conclusion now follows from the lemma □

Corollary 1.0.4 *The statement*

$$\forall\phi[\forall\psi\ \exists b>0\ \forall n\in N|\int_{R_n}\phi\psi|<b \longrightarrow \forall\psi\ \forall b>0\ \forall n\in N|\int_{R_n}\phi\psi|<b]$$

holds in the C^ω-ring classifier.

Proof: Immediate from the previous proposition and the definition of forcing.

Following [2], we let $\mathcal{D}'_c(R^m)$ be the *object of distributions of compact support on* R^m, i.e., the object of R-linear maps $R^{(R^m)} \longrightarrow R$.

Let $\mu \in \mathcal{D}'_c(R^m)$ and let $\phi \in R^{(R^m)}$. We say that ϕ is *a density* of μ iff

$$\forall\psi\ \forall\epsilon>0\ \exists n_0\in N\ \forall n>n_0\ |\mu(\psi)-\int_{R_n}\phi\psi|<\epsilon$$

Notice that we cannot define *the* density since the $\epsilon-\delta$ definition of limit does not define the limit uniquely in our context.

We say that μ is *trivial* if the the function 0 is a density of μ.

Theorem 1.0.5 *Any distribution μ of compact support on R^m having a density is trivial.*

Proof: Let ϕ be a density of μ and let ψ and $\epsilon > 0$ be arbitrary. We claim that $|\mu(\psi)| < \epsilon$. In fact, by the definition of density, there is some $n_0 \in N$ such that for all $n > n_0$

$$|\int_{R_n}\phi\psi-\mu(\psi)|<\epsilon/2 \qquad (*)$$

This implies that the sequence $\{|\int_{R_n}\phi\psi|\}_{n>n_0}$ is bounded by $\epsilon/2 + |\mu(\psi)|$ which in turn implies that the whole sequence $\{|\int_{R_n}\phi\psi|\}_n$ is bounded. By the corollary, this sequence is bounded by every $b > 0$ and this implies that for every n,

$$|\int_{R_n} \phi\psi| < \epsilon/2 \qquad (**)$$

Write $\mu(\psi) = (\mu(\psi) - \int_{R_n} \phi\psi) + \int_{R_n} \phi\psi$. Thus, $|\mu(\psi)| \leq |\mu(\psi) - \int_{R_n} \phi\psi| + |\int_{R_n} \phi\psi|$. Since this holds for every n, the first summand can be made $< \epsilon/2$, by choosing $n > n_0$ (cf. (*)), while the second is $< \epsilon/2$ (by (**)). Thus, $|\mu(\psi)| < \epsilon$ and this trivially implies that 0 is a density of μ □

2 Appendix: Analytic preliminaries

The main result needed from classical analysis is the following probably well-known characterization of continuous functions having compact support:

Proposition 2.0.6 *Let $\phi \in C^0(\mathbb{R}^m)$. Then the following are equivalent*

(1) The function ϕ has compact support

(2) For every $\psi \in C^\omega(\mathbb{R}^m)$ the sequence $\{|\int_{R_n} \phi\psi|\}_n$ is bounded.

Proof: (Suggested by Paul M. Gauthier). Recall that $R_n = [-n, n]^m$. Obviously (1) implies (2). To prove the other direction we proceed by contraposition. Assume that (1) is not true. We define (by recursion) a strictly increasing subsequence $\{D_i\}_i$ of the R_n's and a sequence $\{B_i\}_i$ of disjoint closed m-balls with the following further properties:

(i) $\phi(x) \neq 0$ for all $x \in B_i$

(ii) $B_i \subseteq (D_{i+1} \setminus D_i)$

Assume that we have defined a strictly increasing sequence $D_1, \ldots, D_n$ of R_n's together with a sequence $B_1, \ldots, B_{n-1}$ of pairwise disjoint m-balls satisfying (i) and (ii) for all $i < n$. Let B_n be any ball disjoint from D_n such that $\phi(x) \neq 0$ for all $x \in B_n$. (The existence of such a ball follows from the fact that the support of ϕ is unbounded). Finally, let D_{n+1} be any of the R_n's which contains B_n. It is easy to check that the sequence thus determined satisfies all the required properties. Furthermore $\bigcup_i (D_i \setminus D_{i-1}) = \mathbb{R}^m$.

Let $\{\epsilon_i\}_i$ be a decreasing sequence of positive reals such that for all i

$$\epsilon_i \int_{D_i} |\phi| < 1/2^{i+1} \qquad (*)$$

Claim 1: There is a strictly positive function $\epsilon \in C^0(\mathbb{R}^m)$ with the property that for every i

$$\epsilon|_{(D_i \setminus D_{i-1})} < \epsilon_i \qquad (**)$$

Let $d(x,y)$ be the distance between the points x and y. If $x \in \mathbb{R}^m$, then there is a unique i such that $x \in D_i \setminus D_{i-1}$. The line determined by the origin and x cuts the boundary ∂D_{i-1} of D_{i-1} in two points p and p' such that $d(x,p) < d(x,p')$. Similarly, this line cuts the boundary of D_i in two points q and q' with the property that $d(x,q) < d(x,q')$. Define the value of the function ϵ on x by the formula

$$\epsilon(x) = \epsilon_i + d(x,p)/d(q,p)(\epsilon_{i+1} - \epsilon_i)$$

Notice that if $x \in \partial D_i$, then $\epsilon_i(x) = \epsilon_{i+1}$. Furthermore, if $p \in \partial D_{i-1}$, then $lim_{x \to p} \epsilon(x) = \epsilon_i$. Thus ϵ is continuous and clearly has property $(**)$.

By using "bump" functions, we can construct a sequence $\{f_i\}_i$ of continuous functions with the property that for each i the support of f_i is contained in B_i^0 and

$$\int_{B_i} \phi f_i > 2 \qquad (***)$$

We define f by glueing the f_i's and extending by zero. More precisely, let $B = \bigcup_i B_i$. Define

$$f(x) = \begin{cases} f_i(x) & \text{if } x \in B_i \\ 0 & \text{if } x \notin B \end{cases}$$

Clearly $f \in C^0(\mathbb{R}^m)$. By Whitney's approximation theorem (see eg [5]), there is a $\psi \in C^\omega(\mathbb{R}^m)$ such that for all $x \in \mathbb{R}^m$, $|\psi(x) - f(x)| < \epsilon(x)$.

Claim 2: For all i,

$$\int_{B_i} \phi\psi > 3/2$$

In fact, $\int_{B_i} \phi\psi = \int_{B_i} \phi f + \int_{B_i} \phi(\psi - f)$. The first term is greater than 2, and the absolute value of the last term is bounded by $1/2$:

$$|\int_{B_i} \phi(\psi - f)| \leq \int_{B_i} \epsilon|\phi| \leq \int_{D_{i+1}\setminus D_i} \epsilon|\phi| \leq \epsilon_{i+1} \int_{D_{i+1}} |\phi| < 1/2^{i+1} < 1/2$$

Claim 3: For all i,

$$|\int_{(D_i \setminus D_{i-1})\setminus B} \phi\psi| < 1/2^{i+1}$$

Indeed, since $f = 0$ outside B, the left handside equals

$$|\int_{(D_i \setminus D_{i-1})\setminus B} \phi(\psi - f)|$$

But this term is bounded by

$$\int_{(D_i \setminus D_{i-1})\setminus B} |\phi|\epsilon \leq \epsilon_i \int_{D_i} |\phi| \leq 1/2^{i+1}$$

Putting these things together and using the equality

$$D_i = [\bigcup_{j=2}^{i} (D_j \setminus D_{j-1}) \setminus B] \bigcup D_1 \bigcup [\bigcup_{j=1}^{i-1} B_j]$$

we conclude that

$$|\int_{D_i} \phi\psi| \geq (3/2)(i-1) - (\Sigma_{j=2}^{i} 1/2^{j+1} + |\int_{D_1} \phi\psi|) \geq (3/2)(i-1) - (1 + |\int_{D_1} \phi\psi|)$$

This implies that the sequence $\{|\int_{R_n} \phi\psi|\}_n$ is unbounded. Thus, we have established that (2) is not true, concluding the proof $\Box$

Remark 2.0.7 We formulated this proposition for $R_n = [-n, n]^m$ for simplicity, but is clear that we could have used other R_n's. For instance, it seems that the same proof applies when the R_n's form an increasing sequence of convex compact sets whose union is the whole space. Of course, there could be other choices that presumably would require different proofs. We will not go into these matters here.

References

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