General Relativity: Affine connections, parallel transport and sprays

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21 January 2009

In the following, we will use tacitly two results which in many cases reduce linearity of an application between R-modules to just homogeneity.

Proposition 0.1 *If M is a microlinear space, then all the tangent space of M satisfy the Kock-Lawvere axiom in the sense that the canonical map $\alpha : T_m(M) \times T_m(M) \longrightarrow T_m^D$ given by $\alpha(u,v) = u + dv$ is a bijection for each $m \in M$.*

Proof: (cf. [3] page 186) Let $\phi : D \longrightarrow T_x(M)$ be given and let $u = \phi(0)$. We claim the existence of a unique $v \in T_x(M)$ such that $\phi(d) = u + dv$. By replacing $\phi(d)$ by $\phi(d) - u$ in $T_x(M)$ we may assume that $\phi(0)$ is the null vector in $T_x(M)$. Define $\tau : D \times D \longrightarrow M$ by $\tau(d_1, d_2) = \phi(d_1)(d_2)$. Since M is microlinear and

$$D \times D \underset{p_2}{\overset{p_1}{\rightrightarrows}} D \times D \xrightarrow{\ \cdot\ } D$$

is an R-coequalizer, (here $p_1(d_1, d_2) = (d_1, 0), p_2(d_1.d_2) = (0, d_2)$) and $\tau \circ p_1 = \tau \circ p_2$, there is a unique $v \in T_x(M)$ such that $\tau(d_1, d_2) = v(d_1 \cdot v_2) = d_1 \cdot v(d_2)$. Thus, $\phi(d_1) = d_1 \cdot v_1$, completing the proof.

Proposition 0.2 *Let $H : V \longrightarrow W$ be an arbitrary (not necessarily linear) map of R-modules. Assume that W is a microlinear space which satisfies the Kock-Lawvere axiom in the sense that the canonical map $\alpha : W \times W \longrightarrow W^D$ defined by $\alpha(w_1, w_2)(d) = w_1 + dw_2$ is a bijection. Then H is linear iff H is homogeneous.*

Proof: Let $H : V \longrightarrow W$ be homogeneous, i.e., $H(\alpha v) = \alpha H(v)$ for every $\alpha \in R$ and $v \in V$. Take $v_1, v_2 \in V$ and consider the maps

$$D(2) \overset{\phi}{\underset{\psi}{\rightrightarrows}} W$$

defined by

$$\begin{cases} \phi(d_1, d_2) = H(d_1 v_1 + d_2 v_2) \\ \psi(d_1, d_2) = H(d_1 v_1) + H(d_2 v_2) \end{cases}$$

Since $\phi \circ i_k = \psi \circ i_k$ for $k = 1, 2$ and

$$\begin{array}{ccc} 1 & \xrightarrow{\;0\;} & D \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle i_2} \\ D & \xrightarrow[i_1]{} & D(2) \end{array}$$

is an R-pushout, it follows that $\phi = \psi$. In particular $\phi(d, d) = \psi(d, d)$, i.e. $H(d(v_1 + v_2)) = H(dv_1) + H(dv_2)$. Using homogeneity and the fact that α is an isomorphism, $H(v_1) + H(v_2) = H(v_1 + v_2)$.

1 Connections and Sprays on a Microlinear Space

Let M be a microlinear space and let $K : M^{D \times D} \longrightarrow M^D \times_M M^D$ be the canonical map defined by $K(t)(d) = (t(d, 0), t(0, d))$

An *affine connection on* M is section of K

$$\nabla : M^D \times_M M^D \longrightarrow M^{D \times D}$$

satisfying some homogeneity conditions. More precisely, we require ∇ to have the following properties

$$\begin{cases} \nabla(t_1, t_2)(d_1, 0) = t_1(d_1) \\ \nabla(t_1, t_2)(0, d_2) = t_2(d_2) \\ \nabla(\alpha t_1, t_2)(d_1, d_2) = \nabla(t_1, t_2)(\alpha d_1, d_2) \\ \nabla(t_1, \alpha t_2)(d_1, d_2) = \nabla(t_1, t_2)(d_1, \alpha d_2) \end{cases}$$

A connection is *symmetric or torsion-free* if $\nabla(t_1, t_2)(d_1, d_2) = \nabla(t_2, t_1)(d_2, d_1)$.

There is a simple geometric interpretation of a connection in terms of parallel transport. A *parallel transport on* M is a function which associates with each $(t, h) \in M^D \times D$ a bijection

$$\tau_h(t, -) : \pi^{-1}(t(0)) \longrightarrow \pi^{-1}(t(h))$$

such that

$$\begin{cases} \tau_0(t_1, t_2) = t_2 \\ \tau_h(t_1, \lambda t_2) = \lambda \tau_h(t_1, t_2) \\ \tau_h(\lambda t_1, t_2) = \lambda \tau_h(t_1, t_2) \end{cases}$$

for all $\lambda \in R$.

We say that that $\tau_h(t_1, t_2)$ is the result of transporting t_2 parallel to itself along t_1 during h seconds.

Proposition 1.1 *Assume that* M *is a microlinear space. If* τ *is a parallel transport on* M*, then the map* $\nabla : M^D \times_M M^D \longrightarrow M^{D \times_D}$ *defined by*

$$\nabla(t_1, t_2)(h_1, h_2) = \tau_{h_1}(t_1, t_2)(h_2)$$

is an affine connection on M*. Conversely, any affine connection on* M *determines a parallel transport* τ *defined by the same formula.*

Proof: See [3] page 190.

In terms of parallel transport, the notion of a symmetric connection also has a clear geometrical interpretation that the reader is encouraged to find.

When $M = R^n$, a vector t at $\underline{a} \in R^n$ may be written uniquely as $t(d) = \underline{a} + d\underline{b}$, by using Kock-Lawvere axiom. The same axiom also implies that a connection can be written as

$$\nabla(t_1, t_2)(d_1, d_2) = \underline{a} + \underline{b}_1 d_1 + \underline{b_2} d_2 + \tilde{\nabla}_{\underline{a}}(b_1, b_2) d_1 d_2$$

with $t_i(d) = \underline{a} + \underline{b}_i d$ $(i = 1, 2)$

Thus, ∇ is completely determined by by

$$\tilde{\nabla}_{\underline{a}}(-, -) : R^n \times R^n \longrightarrow R^n$$

The defining properties of ∇ are translated into properties of $\tilde{\nabla}_{\underline{a}}(-, -)$ and express simply that this last map is bilinear. If ∇ is symmetric, this map is also symmetric. Notice, however, that in this case such a map is completely determined by its values at the diagonal. Indeed,

Proposition 1.2 *Let* $\sigma : R^n \times R^n \longrightarrow R^n$ *be a map that is bilinear and symmetric and let* $\delta : R^n \longrightarrow R^n$ *be defined by* $\delta(t) = \sigma(t,t)$*. Then*

$$\sigma(t_1, t_2) = (1/2)[\delta(t_1 + t_2) - \delta(t_1) - \delta(t_2)]$$

Proof: We have $\delta(t_1+t_2) = \sigma(t_1+t_2, t_1+t_2) = \sigma(t_1, t_1)+2\sigma(t_1, t_2)+\sigma(t_2, t_2)$, i.e., $\sigma(t_1, t_2) = (1/2)[\delta(t_1 + t_2) - \delta(t_1) - \delta(t_2)]$.

NB For $n = 1$, i.e., for $\sigma : R \times R \longrightarrow R$, homogeneity implies symmetry. In fact, by homogeneity

$$(1) \qquad \sigma(ax, y) = a\sigma(x, y) = \sigma(x, ay)$$

Taking partial derivatives with respect to a we get

$$(2) \qquad x(\partial\sigma/\partial x)(ax, y) = \sigma(x, y) = y(\partial\sigma/\partial y)(x, ay)$$

Taking partial derivatives with respect to x we get

$$(3) \qquad (\partial\sigma/\partial x)(ax, y) + ax(\partial^2\sigma/\partial^2 x)(ax, y) = (\partial\sigma/\partial x)(x, y).$$

Letting $a = 1$,

$$(4) \qquad x(\partial^2\sigma/\partial^2 x)(x, y) = 0$$

Similarly for y. From (1) and (2),

$$\begin{cases} \sigma(x, 0) = \sigma(0, y) = 0 \\ x(\partial\sigma/\partial x)(x, 0) = 0 \\ y(\partial\sigma/\partial y)(0, y) = 0 \end{cases}$$

Lavendhomme principle implies

$$(5) \qquad \begin{cases} (\partial^2\sigma/\partial x^2)(x, y) = 0 \\ (\partial\sigma/\partial x)(x, 0) = 0 \\ (\partial\sigma/\partial y)(0, y) = 0 \end{cases}$$

By Taylor developing around $(0, 0)$,

$$\sigma(x, y) = cxy$$

where $c = (\partial^2\sigma/\partial x\partial y)(0, 0)$.

Thus, σ is symmetric, $\delta(x) = cx^2$ is a quadratic form and $\sigma(x,y) = 1/2[\delta(x+y) - \delta(x) - \delta(y)]$.

For $n > 1$ homogeneity does not imply symmetry. In fact, already for $n = 2$ the map $f : R^2 \times R^2 \longrightarrow R^2$ defined by $f((x_1, x_2), (y_1, y_2)) = (x_1 y_2, x_2 y_1)$ is homogeneous, but not symmetric.

In the following we prove a generalization of the last proposition.

Let M be a microlinear space. A *spray* on M is a map $\sigma : M^D \longrightarrow M^{D_2}$ satisfying the conditions

$$\begin{array}{lll} (1) & \sigma(t)(d) = t(d) & \text{for all } d \in D \\ (2) & \sigma(\lambda t)(\delta) = \sigma(t)(\lambda\delta) & \text{for all } \delta \in D_2, \lambda \in R \end{array}$$

For $M = R^n$, we easily see that a spray has the following description

$$\sigma(t)(\delta) = \underline{a} + \underline{b}\delta + \tilde{\sigma_a}(\underline{b})\delta^2$$

with the homogeneity condition (2)

$$\tilde{\sigma_a}(\lambda\underline{b}) = \lambda^2 \tilde{\sigma_a}(\underline{b})$$

Proposition 1.3 *Given any affine connection (not necessarily symmetric) ∇ on M there is a unique spray σ on M such that*

$$\sigma(t)(d_1 + d_2) = \nabla(t,t)(d_1, d_2)$$

Proof: This follows from the fact that

$$D \underset{j_2}{\overset{j_1}{\rightrightarrows}} D \times D \xrightarrow{\;+\;} D_2$$

is an R-coequalizer, where $j_1(d) = (d, 0)$ and $j_2(d) = (0, d)$. Indeed, for $t \in M^D$, $\nabla(t,t)(d_1, d_2)$ satisfies $\nabla(t,t)(d,0) = t(d) = \nabla(t,t)(0,d)$. Using the previous R-coequalizer, we conclude that there is a unique map $\sigma(t) : D_2 \longrightarrow M$ satisfying

$$\sigma(t)(d_1 + d_2) = \nabla(t,t)(d_1, d_2)$$

It is easy to check that σ satisfies the conditions to be a spray.

The main result of this section is the following version of the Ambrose-Palais-Singer theorem:

Theorem 1.4 *Let M be a microlinear space. Then there is a natural one-to-one correspondence between symmetric affine connections ∇ on M and sprays on M given by*

$$\sigma(t)(d_1 + d_2) = \nabla(t,t)(d_1, d_2)$$

Proof: cf [3] page 192.

Before starting the proof, we observe that the following are R-coequalizers

$$(i) \qquad D \underset{j_2}{\overset{j_1}{\rightrightarrows}} D \times D \xrightarrow{+} D_2$$

$$(ii) \qquad D(2) \times D(2) \underset{(p_2,p_1)}{\overset{id}{\rightrightarrows}} D(2) \times D(2) \xrightarrow{+} D_2(2)$$

$$(iii) \qquad D \times D \times D \underset{(p_2, m \circ p_{13})}{\overset{(p_1, m \circ p_{23})}{\rightrightarrows}} D \times D \xrightarrow{m} D$$

$$(iv) \qquad D \times D \times D(2) \underset{(p_2, m \circ p_{13})}{\overset{(p_1, m \circ p_{23})}{\rightrightarrows}} D \times D(2) \xrightarrow{m} D(2)$$

$$(v) \qquad D_2 \times D_2 \times D_2(2) \underset{(p_2, m \circ p_{13})}{\overset{(p_1, m \circ p_{23})}{\rightrightarrows}} D_2 \times D_2(2) \xrightarrow{m} D_2(2)$$

where m stands for "multiplication."

From connections to sprays: already done using the R-coequalizer (i) (cf. last proposition)

From sprays to connection: we use the R-coequalizer (v). Let σ be a spray, $t_1, t_2 \in M^D$ such that $t_1(0) = t_2(0) = x$. The map $\tau : D_2 \times D_2(2) \longrightarrow M$ defined by $\tau(\delta, \lambda_1, \lambda_2) = \sigma(\lambda_1 t_1 + \lambda_2 t_2)(\delta)$ equalizes the parallel arrows in (v), by homogeneity of σ. Therefore, there is a unique map $\nabla'(t_1, t_2) : D_2(2) \longrightarrow M$ such that $\nabla'(t_1, t_2)(\delta\lambda_1, \delta\lambda_2) = \tau(\delta, \lambda_1, \lambda_2) = \sigma(\lambda_1 t_1 + \lambda_2 t_2)(\delta)$. Then $\nabla = \nabla'|_{D \times D} : D \times D \longrightarrow M$ is an affine symmetric connection, as can be easily checked. It remains to prove that these operations are inverse of each other: see [3].

2 Vector bundles and vector fields

Let M be a microlinear space. A *vector bundle over* M is a map $p : E \longrightarrow M$ with E microlinear such that every fiber $E_m = p^{-1}(m)$ is equipped with an R-module structure satisfying the Kock-Lawvere axiom, i.e., the canonical map

$$H_m : E_m \times E_m \longrightarrow (E_m)^D$$

defined by $H_m(u,v)(d) = u + dv$ is a bijection.

Examples:

(1) the canonical map $\pi_M : M^D \longrightarrow M$ sending a vector v into $v(0)$ is a vector bundle over M. (This was already proved. Cf. Proposition 0.1)

(2) the canonical map $\pi_E : E^D \longrightarrow E$. We denote the R-module structure by $\oplus$ and $\odot$ (same as (1))

(3) the map $p^D : E^D \longrightarrow M^D$. This is not so obvious. We check that its fibers satisfy the Kock-Lawvere axiom. Indeed, given $\phi : D \longrightarrow (E^D)_t = (p^D)^{-1}(t)$ with $t \in M^D$, consider $\phi_d : D \longrightarrow E_{t(d)} = p^{-1}(t(d))$ defined by $\phi_d(d') = \phi(d')(d)$. Since E is a vector bundle, there is a unique couple $(u_d, v_d) \in E^2_{t(d)}$ such that $\phi(d') = u_d + d'v_d$ $(\forall d' \in D)$. where $u = [d \mapsto u_d]$ and $v = [d \mapsto v_d]$ are in $(E^D)_t$. The R-module structure is defined pointwise and denoted by $+$ and $.$

(4) the first projection $p_1 : E \times_M E \longrightarrow E$. The R-module structure, (again denoted by $\oplus$ and $\odot$) is defined by

$$(u,v) \oplus (u,v') = (u, v + v') \qquad \alpha \odot (u,v) = (u, \alpha v)$$

(5) the canonical map $E \times_M E \longrightarrow M$. The R-module structure is defined by

$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2) \qquad \alpha(u,v) = (\alpha u, \alpha v)$$

(If all E_m's satisfy the Kock-Lawvere axiom, so do all products $E_m \times E_m$.)

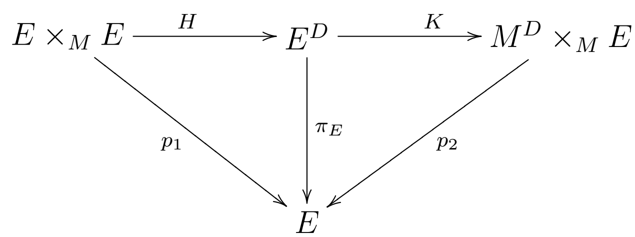
(6) the second projection $p_2 : M^D \times_M E \longrightarrow E$. The corresponding R-structure is given by

$$(t,v) \oplus (t',v) = (t \oplus t', v) \qquad \alpha \odot (t,v) = (\alpha t, v)$$

(7) the first projection $p_1 : M^D \times_M E \longrightarrow M^D$. The corresponding structure is defined by

$$(t, v) + (t, v') = (t, v + v') \qquad \alpha(t, v) = (t, \alpha v)$$

Whenever $p : E \longrightarrow M$ is a vector bundle, we have the following diagram



where $H(u, v)(d) = u + dv$ and $K(t) = (p \circ t, t(0))$.

Proposition 2.1 *1. H is linear as a map of vector bundles (4) and (2)*

2. K is linear as a map of vector bundles (2) and (6)

3. The sequence of linear maps over E

$$0 \longrightarrow E \times_M E \xrightarrow{\ H\ } E^D \xrightarrow{\ K\ } M^D \times_M E$$

is exact, i.e., H is 1-1 and $Im(H) = Ker(K)$.

Proof: (1) and (2). Linearity follows from homogeneity, but this is obvious in each case. (3) H is monic is a consequence of the fact that the fibers of $p : E \longrightarrow M$ satisfy the Kock-Lawvere axiom. It remains to show that $Im(H) = Ker(K)$. Let $m \in M$ and $E_m = p^{-1}(m)$ the fiber of p over m. Then

$$\begin{aligned} t \in E_m^D &\leftrightarrow \forall d \in D\ t(d) \in E_m \\ &\leftrightarrow \forall d \in D\ p(t(d)) = m \\ &\leftrightarrow p \circ t = 0_m \\ &\leftrightarrow t \in Ker(K) \end{aligned}$$

But, by definition of vector bundle, $t \in E_m^D$ iff there is a unique couple $(u, v) \in E_m^2$ such that $\forall d \in D\ t(d) = u + dv$, i.e., if $t = H(u, v)$. This finishes the proof.

A vector $t \in E^D$ such that $K(t) = 0$ is called a *vertical vector.*

Our aim is to generalize the definition of affine connections to vector bundles, and introduce the connection maps.

We recall some notions to be used in the sequel (cf. [3]and [2]).

Let M be a micro-linear space and let $p : E \longrightarrow M$ be a vector bundle. An *affine connection* on $p : E \longrightarrow M$ (briefly, on E) is a section of K

$$\nabla : M^D \times_M E \longrightarrow E^D$$

which is linear with respect to both vector bundles structures $\oplus$ over E and $+$ over M^D. Recall that

$$\begin{cases} (u,v) \oplus (u,v') = (u, v+v'), & \alpha \odot (u,v) = (u, \alpha v) \\ (u_1,v_1) + (u_2,v_2) = (u_1+u_2, v_1+v_2), & \alpha \cdot (u,v) = (\alpha u, \alpha v) \end{cases}$$

In other words, ∇ should satisfy the following identities

$$\begin{gathered} p \circ \nabla(t,v)(d) = t(d), \nabla(t,v)(0) = v \\ \nabla(\alpha t, v)(d) = (\alpha \odot \nabla(t,v))(d) = \nabla(t,v)(\alpha d) \\ \nabla(t, \alpha v)(d) = (\alpha \cdot \nabla(t,v))(d) = \alpha \cdot (\nabla(t,v)(d)) \end{gathered}$$

Let $\nabla : M^D \times_M E \longrightarrow E^D$ be an affine connection on a vector bundle $p : E \longrightarrow M$. The connection ∇ induces a split-exact sequence of vector bundles over E (i.e. with respect to $\oplus$):

$$0 \hookrightarrow E \times_M E \xrightarrow{H} E^D \underset{\nabla}{\overset{K}{\rightleftarrows}} M^D \times_M E \twoheadrightarrow 0$$

Hence H splits as well. Indeed, there is a map $C_1 : E^D \longrightarrow E \times_M E$ such that

$$\begin{cases} C_1 \circ H = id_{E \times_M E} \\ H \circ C_1 = id_{E^D} \ominus \nabla K \end{cases}$$

Indeed, $K(t \ominus \nabla K(t)) = K(t) - K(t) = 0$. Thus, $t \ominus \nabla K(t) = H(u,v)$ for a unique couple $(u,v) \in E^2$. Define $C_1(t) = (u,v)$. Then $C_1 \circ H = id_{E \times_M E}$ and $H \circ C_1 = id_{E^d} \ominus \nabla K$.

We define *the connection map associated to* ∇ to be

$$C : E^D \longrightarrow E$$

given by $C = p_2 \circ C_1$.

The connection map $C : E^D \longrightarrow E$ is bilinear in the sense that for every $(t, t') \in E^D \times_{M^D} E^D$, we have

$$\begin{cases} C(t + t') = C(t) + C(t') \\ C(t \oplus t') = C(t) + C(t') \end{cases}$$

(The two +'s on the right denote addition in $E_{pt(0)} = E_{pt'(0)}$).

Intuitively, C measures the extent to which a tangent vector $t \in E^D$ coincides with the parallel transport of $t(0)$ along $p \circ t$ given by ∇. (Cf. [3] V3.9-11 and V4.7)

We shall apply these notions to the vector bundle $E = M^D$ (and $p = \pi_0$). So, we assume that $M = E^D$. If X, Y are vector fields, we define the new vector field

$$\nabla_X Y = C \circ (X \star Y)$$

with $(X \star Y)(d_1, d_2) = Y_{d_2} \circ X_{d_1}$. We notice that $X \star Y$ is a transpose of the map $Y^D \circ X : M \longrightarrow (M^D)^D$. Finally, we define *the curvature of the connection* ∇ to be the map

$$R_{XY} Z = (C \circ C^D - C \circ C^D \circ \Sigma) \circ (Z \star Y \star X)$$

where

$$\begin{cases} (Z \star Y \star X)(d_1, d_2, d_3) = Z_{d_3} \circ Y_{d_2} \circ X_{d_1} \\ \Sigma(Z \star Y \star X)(d_1, d_2, d_3) = Z_{d_3} \circ Y_{d_1} \circ X_{d_2} \end{cases}$$

Notice that $Z \star Y \star X$ is a transpose of $(Z^D)^D \circ Y^D \circ X : M \longrightarrow ((M^D)^D)^D$.

There is a geometrical interpretation of $\nabla_X Y$:

Proposition 2.2 *Let $E \longrightarrow M$ be a vector bundle, ∇ an affine connection on E, X a vector field on M and Y an E-vector field on M. Then $\nabla_X Y$ is uniquely determined by the following identity*

$$\forall h \in D \; h(\nabla_X Y)_m = \nabla(d \mapsto \tilde{X}_m(h + d), Y^D \circ X(h))(-h) - Y_m$$

where $\tilde{X} : M \longrightarrow M^{D_2}$ is the (unique) extension of X.

Proof: (Almost verbatim from [3]). We first notice that H is a bijection (since E_m satisfies the Kock-Lawvere axiom) and this implies that

$$\begin{aligned} H \circ C_1(t)(h) &= p_1 \circ H^{-1} H \circ C_1(t) + h p_2 H^{-1} H C_1(t) \\ &= H \circ C_1(t)(0) + hC(t) \end{aligned}$$

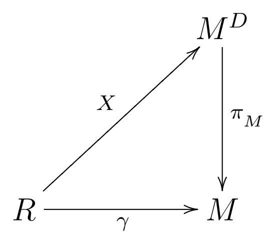
In particular, for $t = Y^D \circ X - \nabla(X, Y)$ we have

$$\forall h \in D \; [(Y^D \circ X)_m - \nabla(X_m, Y_m)](h) = Y_m + h(\nabla_X Y)_m]$$

The rest of the proof is straightforward and may be found in [3].

2.1 Vector fields over curves

Let M be a microlinear space and $\gamma : R \longrightarrow M$. A *vector space over* γ is a map $X : R \longrightarrow M^D$ such that the diagram



is commutative.

Example: the "velocity field" of γ, $\gamma^\bullet : R \longrightarrow M^D$ defined by $\gamma^\bullet(t)(d) = \gamma(t + d)$.

If ∇ is an affine connection on M and X a vector field over γ, we define $\nabla_{\gamma^\bullet} X$, a new vector field over γ, by the equation, for all $h \in D$,

$$h \nabla_{\gamma^\bullet} X(t) = \nabla(\gamma^\bullet(t), X(\gamma(t + h))(-h) - X(\gamma(t))$$

(Notice that existence and uniqueness of $\nabla_{\gamma^\bullet} X(t)$ follow from the validity of Kock-Lawvere axiom for the fiber $M_{\gamma(t)}$.) For the properties of this operation, see [5].

2.2 Computations for $M = R^n$

Proposition 2.3 (e.g. Fock page 128) *If* $X = \sum_i a^i \partial/\partial x^i$ *and* $Y = \sum_i b^i \partial/\partial x^i$ *are two vector fields on* $M = R^n$*, then*

$$\nabla_X Y = \sum_{ij} a^i (\partial b^j/\partial x^i) \partial/\partial x^j + \sum_{ijk} \Gamma^k_{ij} a^i b^j \partial/\partial x^k$$

In particular the k^{th}*-component of* $L_X Y$ *is*

$$(\nabla_X Y)^k = \sum_i a^i (\partial b^k/\partial x^i) + \sum_{ij} \Gamma^k_{ij} a^i b^j$$

Proof: We first prove the particular case $X = \partial/\partial x^i$:

$$\begin{aligned} \nabla_{\partial/\partial x^i} Y &= \nabla_{\partial/\partial x^i} \textstyle\sum_j b^j \partial/\partial x^j \\ &= \textstyle\sum_j \nabla_{\partial/\partial x^i} b^j \partial/\partial x^j \\ &= \textstyle\sum_j ((\partial b^j/\partial x^i) \partial/\partial x^j + b^j \nabla_{\partial/\partial x^i} \partial/\partial x^j) \\ &= \textstyle\sum_j ((\partial b^j/\partial x^i) \partial/\partial x^j + b^j \sum_k \Gamma^k_{ij} \partial/\partial x^k) \end{aligned}$$

The general case follows from the properties of $\nabla_X Y$ (cf. Lavendhomme [2] page 147). In fact,

$$\begin{aligned} \nabla_X Y &= \nabla_{\sum_i a^i \partial/\partial x^i} Y \\ &= \textstyle\sum_i \nabla_{a^i \partial/\partial x^i} Y \\ &= \textstyle\sum_i a^i \nabla_{\partial/\partial x^i} Y \\ &= \textstyle\sum_i a^i [\sum_j ((\partial b^j/\partial x^i) \partial/\partial x^j + b^j \sum_k \Gamma^k_{ij} \partial/\partial x^k)] \\ &= \textstyle\sum_{ij} a^i (\partial b^j/\partial x^i) \partial/\partial x^j + \sum_{ijk} a^i b^j \Gamma^k_{ij} \partial/\partial x^k \end{aligned}$$

3 Geodesics

Proposition 3.1 *Let* $\gamma : R \longrightarrow M$ *and* ∇ *a connection on* M*. Then the following are equivalent:*

1. $\nabla(\gamma^\bullet, \gamma^\bullet) = \gamma^{\bullet\bullet}$
2. $\nabla_{\gamma^\bullet} \gamma^\bullet = 0$

Proof: Unwinding definitions and writing everything explicitly, these expressions are shorthands for

1. $\gamma^\bullet(t+h_1) = \tau_{h_1}(\gamma^\bullet(t), \gamma^\bullet(t))$
2. $\tau_{-h}(\gamma^\bullet(t+h), \gamma^\bullet(t+h)) = \gamma^\bullet(t)$

We obtain the second from the first by substituting

$$\begin{cases} h_1 \mapsto -h \\ t \mapsto t+h \end{cases}$$

To obtain the other direction, we make the substitution

$$\begin{cases} h \mapsto -h_1 \\ t \mapsto t+h_1 \end{cases}$$

We say that $\gamma : R \longrightarrow M$ is a *geodesic* iff γ satisfies any of the conditions of the proposition.

Thus, geodesic curves have a clear geometrical content: "by parallel transporting an infinitesimal portion of the curve (seen as a tangent vector) along itself, the result is again an infinitesimal portion of the curve, i. e, a new tangent vector to the curve." No metrics are needed to define this notion.

Let M is an n-dimensional manifold. By introducing x-coordinates, we may assume that $M = R^n$, and use Tensor Calculus ("les débauches d'indices" according to Élie Cartan). In this case,

$$\nabla : M \times R^n \longrightarrow M \times R^n \times R^n$$

is completely determined by by its last component ∇_4, since K is a projection and we define the *Chritoffel symbols of the second kind* by

$$(\Gamma^l_{ji})_x(p) = -\nabla_4(p, e_i, e_j)(e_l)$$

We shall usually omit 'x' from the notation, if the coordinate frame is clear from the context.

Lemma 3.2 *If ∇ is a connection, ∇^4 is bilinear.*

Proof: Immediate from the properties $\nabla(\lambda b_1, b_2)(d_1, d_2) = \nabla(b_1, b_2)(\lambda d_1, d_2)$ and $\nabla(b_1, \lambda b_2)(d_1, d_2) = \nabla(b_1, b_2)(d_1, \lambda d_2)$.

Proposition 3.3 *A geodesics $\gamma = (x^1, \ldots, x^n)$ satisfies the second order differential equation*

$$d^2x^k/dt^2 + \sum_{i,j=1}^{n} \Gamma^k_{\alpha\beta}(dx^\alpha/dt)(dx^\beta/dt) = 0$$

Proof: Recall that the vector field along γ, $\gamma^\bullet : R \longrightarrow M^D$ is defined by $\gamma^\bullet(t) = [d \mapsto \gamma(t+d)]$, i.e., $\gamma^\bullet(t)(d) = \gamma(t+d)$. It follows that $\gamma^{\bullet\bullet} : R \longrightarrow (M^D)^D$ is given by $\gamma^{\bullet\bullet}(t)(d_1, d_2) = \gamma(t + d_1 + d_2)$. Hence

$$\begin{aligned} \gamma^\bullet(t)(d) &= (x^1(t+d), \ldots, x^n(t+d)) \\ &= (x^1(t) + d(dx^1/dt)(t)), \ldots, x^n(t) + d(dx^n/dt(t))) \\ &= \underline{x}(t) + d(d\underline{x}/dt)(t) \end{aligned}$$

with $\underline{x} = (x^1(t), \ldots, x^n(t))$ and $d\underline{x}/d(t) = (dx^1/dt(t), \ldots, dx^n/dt(t))$. Leaving the "$t$" out,

$$\begin{aligned} \gamma^{\bullet\bullet}(d_1, d_2) &= (x^1 + d_1(dx^1/dt) + d_2(dx^1/dt)) + d_1d_2(d^2x^1/dt^2), \ldots, \\ &\quad x^n + d_1(dx^n/dt) + d_2(dx^n/dt) + d_1d_2(d^2x^n/dt^2)) \\ &= \underline{x} + d_1(d\underline{x}/dt) + d_2(d\underline{x}/dt) + d_1d_2(d^2\underline{x}/dt^2) \end{aligned}$$

Similarly,

$$\nabla(\gamma^\bullet, \gamma^\bullet)(d_1, d_2) = \underline{x} + d_1(d\underline{x}/dt) + d_2(d\underline{x}/dt) + d_1d_2\nabla^4_{\underline{x}}(d\underline{x}/dt, d\underline{x}/dt)$$

The equation of the geodesic becomes then

$$(*) \quad d^2\underline{x}/dt^2 = \nabla^4_{\underline{x}}(d\underline{x}/dt, d\underline{x}/dt)$$

Writing $d\underline{x}/dt = \sum_\alpha (dx^\alpha/dt)\partial/\partial x^\alpha$ and using the bi-linearity of ∇^4 we have

$$\nabla^4(d\underline{x}/dt, d\underline{x}/dt) = \sum_{\alpha\beta}(dx^\alpha/dt)(dx^\beta/dt)\nabla^4(\partial/\partial x^\alpha, \partial/\partial x^\beta)$$

Taking the k^{th} component in the equation (*), we finally obtain

$$d^2x^k/dt^2 + \sum_{\alpha\beta} \Gamma^k_{\alpha\beta}(dx^\alpha/dt)(dx^\beta/dt) = 0$$

This is the expression found in texts of Differential Geometry (See e,g, [6]).

Proposition 3.4 *Given an affine connection ∇ on M, there is a symmetric affine connection $\tilde{\nabla}$ having the same geodesics.*

Proof: Let ∇ be an affine connection and let $\sigma : M^D \longrightarrow M^{D_2}$ be the unique spray associated to ∇, i.e., defined by the formula $\sigma(t)(d_1+d_2)) = \nabla(t,t)(d_1,d_2)$. Define $\tilde{\nabla}$ to be the symmetric affine connection associated with σ (given by the Ambrose-Palais-Singer Theorem). We claim that ∇ and $\tilde{\nabla}$ have the same geodesics. In fact, if $\gamma : R \longrightarrow M$ is a curve,

$$\begin{array}{rcl} \tilde{\nabla}(\gamma^\bullet(t),\gamma^\bullet(t))(d_1,d_2) & = & \sigma(\gamma^\bullet(t)(d_1+d_2) \\ & = & \nabla(\gamma^\bullet(t),\gamma^\bullet(t))(d_1,d_2) \end{array}$$

(the first by definition of $\tilde{\nabla}$ (cf. Ambrose-Palais-Singer theorem already mentioned); the second by definition of σ). But γ is a geodesic for a connection ∇ iff $\nabla(\gamma^\bullet(t),\gamma^\bullet(t)) = \gamma^{\bullet\bullet}(t)$. Thus, ∇ and $\tilde{\nabla}$ have the same geodesics.

This shows that, so far as geodesics of a connection are concerned, we may assume, without loss of generality, that the connection is symmetric.

4 Transformation law for vectors and vector fields

In this section we study how vectors transform from x-coordinates to x'-coordinates.

Let $\phi = (\phi^1,\ldots,\phi^n) : R^n \longrightarrow R^n$ be a diffeomorphism and $v = (v^1,\ldots,v^n) : D \longrightarrow R^n$ a tangent vector. We want to find the the law of transformation of the components of v under ϕ. Assume that $v^i(d) = a^i + db^i$ $(i = 1,\ldots,n)$. We have

$$D \xrightarrow{\;v\;} R^n \xrightarrow{\;\phi\;} R^n$$

$$(\phi\circ v)(d) = (\phi^1(a) + d\sum_j b_j(\partial\phi^1/\partial x^j)(a),\ldots,\phi^n(a) + d\sum_j b_j(\partial\phi^n/\partial x^j)(a))$$

This means that the i^{th}-component of $\phi\circ v$ is

$$(\phi\circ v)^i = \sum_j b^j(\partial\phi^i/\partial x^j).$$

Define $v'^i = \pi_i(\phi \circ v)$. By identifying v^i with b^i and letting $x'^i(x) = \phi^i(x)$, as is common in texts of GR, our previous expression can be written

$$v'^i = \sum_j v^j(\partial x'^i/\partial x^j)$$

This is the transformation rule "from the x variables to the x' variables". Using a terminology used extensively, vectors transform "contravariantly." Clearly this argument gives the law of transformation for vector fields by just interpreting v'^i and v^j as components of a vector field.

There is a corresponding transformation law for co-vectors which can be deduced from this one, namely

$$\partial/\partial x^\mu = \sum_\alpha (\partial \xi^\alpha/\partial x^\mu)\partial/\partial \xi^\alpha$$

4.1 Transformation law for Γ^k_{ij}

In this section, we present the proof of Synge/Schild [7] of the transformation rule for Γ^k_{ij} when we pass from x-coordinates to x'-coordinates:

$$\begin{array}{rcl}(\Gamma^\gamma_{\alpha\beta})_{x'} & = & \sum_{ijk}(\Gamma^k_{ij}) + x(\partial x'^\gamma/\partial x^k)(\partial x^i/\partial x'^\alpha)(\partial x^j/\partial x'^\beta) \\ & + & \sum_k(\partial^2 x^k/\partial x'^\alpha \partial x'^\beta)(\partial x'^\gamma/\partial x^k)\end{array}$$

(cf. e.g. [6] II page 232 or [7] page 48).

Start with a curve $\gamma : R \longrightarrow M$. Then we know (previous section) that $V = \gamma^{\bullet\bullet} + \nabla(\gamma^\bullet, \gamma^\bullet)$ is a contravariant vector field. In x coordinates, the γ-th component of V is

$$v^\gamma = d^2x^\gamma/dt^2 + \sum_{\alpha\beta} \Gamma^\gamma_{\alpha\beta}(dx^\alpha/du)(dx^\beta/du)$$

Thus, by the transformation law of contravariant vector fields,

$$(2.502) \quad v'^\gamma = \sum_k v^k(\partial x'^\gamma/\partial x^k)$$

Lemma 4.1 (2.504)

$$(d^2x^k/dt^2)(\partial x'^\gamma/\partial x^k) = (d^2x'^\gamma/dt^2) + \sum_{\alpha\beta}(\partial x'^\gamma/\partial x^k)(\partial^2 x^k/\partial x'^\alpha \partial x'^\beta)(dx'^\alpha/dt)(dx'^\beta/dt)$$

Proof: By Leibniz's rule,

$$\begin{array}{rcl} d^2x^k/dt^2 & = & d/dt[(\sum_\alpha \partial x^k/\partial x'^\alpha)(dx'^\alpha/dt)] \\ & = & \sum_\alpha(\partial x^k/\partial x'^m)(d^2x'^m/dt^2) + \sum_{\alpha\beta}(\partial^2 x^k/\partial x'^\alpha \partial x'^\beta)(dx'^\alpha/dt)(dx'^\beta)dt \end{array}$$

Multiplying both sides by $\partial x'^\gamma/\partial x^k$ we obtain the result sought.

Corollary 4.2 (Transformation law for Γ)

$$\begin{array}{rcl} (\Gamma^\gamma_{\alpha\beta})_{x'} & = & \sum_{ijk}(\Gamma^k_{ij})_x(\partial x'^\gamma/\partial x^k)(\partial x^i/\partial x'^\alpha)(\partial x^j/\partial x'^\beta) \\ & + & \sum_k(\partial^2 x^k/\partial x'^\alpha \partial x'^\beta)(\partial x'^\gamma/\partial x^k) \end{array}$$

Proof: From (2.502)

$$\begin{array}{l} d^2x'^\gamma/dt^2 + (\Gamma^\gamma_{\alpha\beta})_{x'}(dx'^\alpha/dt)(dx'^\beta/dt) = \\ \sum_k(d^2x^k/dt^2 + (\Gamma^k_{\alpha\beta})_x(dx^\alpha/dt)(dx^\beta/dt))(\partial x'^\gamma/\partial x^k)) \end{array}$$

Using 2.504, changing bound variables α, β by i, j and noticing that

$$\left\{ \begin{array}{rcl} dx^i/dt & = & \sum_\alpha(\partial x^i/\partial x'^\alpha)(dx'^\alpha/dt) \\ dx^j/dt & = & \sum_\alpha(\partial x^j/\partial x'^\alpha)(dx'^\alpha/dt) \end{array} \right.$$

we obtain

$$A^\gamma_{\alpha\beta}(dx'^\alpha/dt)(dx'^\beta/dt) = 0$$

with

$$\begin{array}{rcl} A^\gamma_{\alpha\beta} & = & (\Gamma'^\gamma_{\alpha\beta})_x - \sum_{ijk}(\Gamma^k_{ij})_x(\partial x'^\gamma/\partial x^k)(\partial x^i/\partial x'^\alpha)(\partial x^j/\partial x'^\beta) \\ & - & \sum_k(\partial x'^\gamma/\partial x^k)(\partial^2 x^k/\partial x'^\alpha \partial x'^\beta) \end{array}$$

Now, $A^\gamma_{\alpha\beta}$ is a function on M, independent of both dx'^α/dt and dx'^β/dt. Therefore, $A^\gamma_{\alpha\beta} = 0$. In more details, let $p \in M$. Let γ be a geodesic starting at p with initial velocity $(1, \ldots, 1)$ in the x'-coordinates. Then $A^\gamma_{\alpha\beta}(p) = 0$. We have obtained the transformation law.

4.2 Transformation law for Γ (bis)

Recall from [5] (page 4) that

$$(*)\ (\nabla_{\partial/\partial x^i}\partial/\partial x^j)_x = \sum_{k=1}^n \Gamma^k_{ij}(x)\partial/\partial x^k|_x$$

By the properties of ∇, if $X = \sum_i a^i \partial/\partial x^j$ and $Y = \sum_j b^j \partial/\partial x^i$, then ([5] page 5)

$$(**)\ \nabla_X Y = \sum_{ij} a^i(\partial b^j/\partial x^i)\partial/\partial x^j + \sum_{ijk} \Gamma^k_{ij} a^i b^j \partial/\partial x^k$$

In x' coordinates, $(*)$ becomes

$$(*')\ (\nabla_{\partial/\partial x'^\alpha} \partial/\partial x'^\beta)_{x'} = \sum_\gamma \Gamma^\gamma_{\alpha\beta}(x')\partial/\partial x'^\gamma|_{x'}$$

and the $\Gamma' s$ are determined by this equation, since $\{\partial/\partial x'^\gamma|_{x'}\}_\gamma$ is a basis.

For the particular case

$$\begin{cases} X = \partial/\partial x'^\alpha = \sum_i (\partial x^i/\partial x'^\alpha)\partial/\partial x^i \\ Y = \partial/\partial x'^\beta = \sum_j (\partial x^j/\partial x'^\beta)\partial/\partial x^j \end{cases}$$

$a^i = \partial x^i/\partial x'^\alpha$ and $b^j = \partial x^j/\partial x'^\beta$ and

$$(***)\ \nabla_{\partial/\partial x'^\alpha} \partial/\partial x'^\beta = \sum_{ij} (\partial x^i/\partial x'^\alpha)\partial/\partial x^j + \sum_{ijk} \Gamma^k_{ij}(\partial x^i/\partial x'^\alpha)(\partial x^j/\partial x'^\beta)\partial/\partial x^k$$

From

$$\partial/\partial x^k = \sum_\sigma (\partial x'^\sigma/\partial x^k)\partial/\partial x'^\sigma$$

we conclude that the γ^{th} term of the second summand of $(***)$ is

$$\sum_{ijk} \Gamma^k_{ij}(\partial x^i/\partial x'^\alpha)(\partial x^j/\partial x'^\beta)(\partial x'^\gamma/\partial x^k)$$

Similarly, from

$$\partial/\partial x^j = \sum_\sigma (\partial x^\sigma/\partial x^j)\partial/\partial x'^\sigma$$

we conclude that the γ^{th} term of the first summand is

$$\begin{aligned} &\textstyle\sum_{ij} (\partial x^i/\partial x'^\alpha)\partial/\partial x^i(\partial x^j/\partial x'^\beta)(\partial x'^\gamma/\partial x^j) \\ &= \textstyle\sum_{ij} \partial x^i/\partial x'^\alpha \sum_\sigma (\partial x'^\sigma/\partial x^i)\partial/\partial x'^\sigma(\partial x^j/\partial x'^\beta)(\partial x'^\gamma/\partial x^j) \\ &= \textstyle\sum_{ij\sigma} (\partial x^i/\partial x'^\alpha)(\partial x'^\sigma/\partial x^i)(\partial^2 x^j/\partial x'^\sigma \partial x'^\beta)(\partial x'^\gamma/\partial x^j) \\ &= \textstyle\sum_{ij\sigma} \delta^\sigma_\alpha(\partial^2 x^j/\partial x'^\sigma \partial x'^\beta)(\partial x'^\gamma/\partial x^j) \\ &= \textstyle\sum_{ij} (\partial^2 x^j/\partial x'^\alpha \partial x'^\beta)(\partial x'^\gamma/\partial x^j) \\ &= \textstyle\sum_i (\partial^2 x^i/\partial x'^\alpha \partial x'^\beta)(\partial x'^\gamma/\partial x^i) \end{aligned}$$

This gives the law of transformation

$$(\Gamma^\gamma_{\alpha\beta})_{x'} = \sum_{ijk} (\Gamma^k_{ij})_x(\partial x^i/\partial x'^\alpha)(\partial x^j/\partial x'^\beta)(\partial x'^\gamma/\partial x^k) + \sum_i (\partial^2 x^i/\partial x'^\alpha \partial x'^\beta)(\partial x'^\gamma/\partial x^i)$$

4.3 A note on the equivalence between definitions of tangent vectors in classical DF and SDG

Recall that classically, a tangent vector at p is defined as a linear derivation at p

$$V_p : C^\infty(U) \longrightarrow R$$

where $p \in U$ and U is an open subset of M. (NB: this definition is independent of U CHECK!)

On the other hand, a tangent vector at p in SDG is simply a map $v_p : D \longrightarrow M$ with $v_p(0) = p$.

Proposition 4.3 *If M is a smooth n-dimensional manifold, there is a one-to-one correspondence between*

$$\frac{\frac{R\text{-linear derivations at } p}{}}{\text{Maps } v : D \longrightarrow M \text{ with } v(0) = p}$$

Proof: We may assume as well that $M = R^n$ and $p = 0$. This proposition is a consequence of Hadamard's. Let V_p be a R-linear derivation at p and $f : U \longrightarrow R$ with $0 \in U$ and U convex open in M. By Hadamard's

$$f(x) = f(0) + \sum_i x^i g_i(x)$$

Since $V_p(c) = 0$ (as easily checked), $V_0(f) = V_0(\sum_i x^i g_i(x))$. Hence,

$$\begin{aligned} V_0(f) &= V_0(\textstyle\sum_i x^i g_i(x)) \\ &= \textstyle\sum_i V_0(x^i g_i(x)) \\ &= \textstyle\sum_i g_i(0) V_0(x^i) + 0 \\ &= \textstyle\sum_i \partial f / \partial x^i(0) V_0(x_i) \\ &= (\textstyle\sum_i a_i \partial / \partial x^i|_0)(f) \end{aligned}$$

Thus the operator V_0 is completely determined by the n-tuple $(a_1, \ldots, a_n)$ (since the operators $\partial/\partial x^i|_0$'s are clearly linearly independent)

By Kock-Lawvere, a map $v_0 : D \longrightarrow R^n$ is of the form $v_0(d) = (a_1 d, \ldots, a_n d)$ for unique n-tuple $(a_1, \ldots, a_n)$. Thus the correspondence can be described as follows: if V_0 is given, we let $v_0(d) = \sum_i V_0(x^i)d$. Conversely, if v_0 is given, say, $v_0(d) = (a_1 d, \ldots, a_n d)$, we let $V_0 = \sum_i a_i \partial/\partial x^i|_0$.

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