In the following, we will use tacitly two results which in many cases reduce linearity of an application between \( R \)-modules to just homogeneity.

**Proposition 0.1** If \( M \) is a microlinear space, then all the tangent space of \( M \) satisfy the Kock-Lawvere axiom in the sense that the canonical map \( \alpha : T_m(M) \times T_m(M) \longrightarrow T^D_m \) given by \( \alpha(u,v) = u + dv \) is a bijection for each \( m \in M \).

**Proof:** (cf. [3] page 186) Let \( \phi : D \longrightarrow T_x(M) \) be given and let \( u = \phi(0) \). We claim the existence of a unique \( v \in T_x(M) \) such that \( \phi(d) = u + dv \). By replacing \( \phi(d) \) by \( \phi(d) - u \) in \( T_x(M) \) we may assume that \( \phi(0) \) is the null vector in \( T_x(M) \). Define \( \tau : D \times D \longrightarrow M \) by \( \tau(d_1,d_2) = \phi(d_1)(d_2) \). Since \( M \) is microlinear and

\[
D \times D \xrightarrow{p_1} D \times D \xrightarrow{p_2} D
\]

is an \( R \)-coequalizer, (here \( p_1(d_1,d_2) = (d_1,0), p_2(d_1,d_2) = (0,d_2) \)) and \( \tau \circ p_1 = \tau \circ p_2 \), there is a unique \( v \in T_x(M) \) such that \( \tau(d_1,d_2) = v(d_1 \cdot v_2) = d_1 \cdot v(d_2) \). Thus, \( \phi(d_1) = d_1 \cdot v_1 \), completing the proof.

**Proposition 0.2** Let \( H : V \longrightarrow W \) be an arbitrary (not necessarily linear) map of \( R \)-modules. Assume that \( W \) is a microlinear space which satisfies the Kock-Lawvere axiom in the sense that the canonical map \( \alpha : W \times W \longrightarrow W^D \) defined by \( \alpha(w_1,w_2)(d) = w_1 + dw_2 \) is a bijection. Then \( H \) is linear iff \( H \) is homogeneous.
Proof: Let $H : V \rightarrow W$ be homogeneous, i.e., $H(\alpha v) = \alpha H(v)$ for every $\alpha \in R$ and $v \in V$. Take $v_1, v_2 \in V$ and consider the maps

$$D(2) \xrightarrow{\phi} W$$

defined by

$$\begin{align*}
\phi(d_1, d_2) &= H(d_1 v_1 + d_2 v_2) \\
\psi(d_1, d_2) &= H(d_1 v_1) + H(d_2 v_2)
\end{align*}$$

Since $\phi \circ i_k = \psi \circ i_k$ for $k = 1, 2$ and

$$
\begin{array}{ccc}
1 & 0 & D \\
\downarrow & & \downarrow i_2 \\
0 & & D \\
\downarrow i_1 & & \downarrow \text{D(2)}
\end{array}
$$

is an $R$-pushout, it follows that $\phi = \psi$. In particular $\phi(d, d) = \psi(d, d)$, i.e.

$H(d(v_1 + v_2)) = H(d v_1) + H(d v_2)$. Using homogeneity and the fact that $\alpha$ is an isomorphism, $H(v_1) + H(v_2) = H(v_1 + v_2)$.

1 Connections and Sprays on a Microlinear Space

Let $M$ be a microlinear space and let $K : M^{D \times D} \longrightarrow M^D \times_M M^D$ be the canonical map defined by $K(t)(d) = (t(d, 0), t(0, d))$

An affine connection on $M$ is section of $K$

$$\nabla : M^D \times_M M^D \longrightarrow M^{D \times D}$$

satisfying some homogeneity conditions. More precisely, we require $\nabla$ to have the following properties

$$\begin{align*}
\nabla(t_1, t_2)(d_1, 0) &= t_1(d_1) \\
\nabla(t_1, t_2)(0, d_2) &= t_2(d_2) \\
\nabla(\alpha t_1, t_2)(d_1, d_2) &= \nabla(t_1, t_2)(\alpha d_1, d_2) \\
\nabla(t_1, \alpha t_2)(d_1, d_2) &= \nabla(t_1, t_2)(d_1, \alpha d_2)
\end{align*}$$

A connection is *symmetric or torsion-free* if

\[ \nabla(t_1, t_2)(d_1, d_2) = \nabla(t_2, t_1)(d_2, d_1) \]

There is a simple geometric interpretation of a connection in terms of parallel transport. A *parallel transport on \( M \)* is a function which associates with each \((t, h) \in M^D \times D\) a bijection

\[
\tau_h(t, -) : \pi^{-1}(t(0)) \to \pi^{-1}(t(h))
\]

such that

\[
\begin{align*}
\tau_0(t_1, t_2) &= t_2 \\
\tau_h(t_1, \lambda t_2) &= \lambda \tau_h(t_1, t_2) \\
\tau_h(\lambda t_1, t_2) &= \lambda \tau_h(t_1, t_2)
\end{align*}
\]

for all \( \lambda \in R \).

We say that that \( \tau_h(t_1, t_2) \) is the result of transporting \( t_2 \) parallel to itself along \( t_1 \) during \( h \) seconds.

**Proposition 1.1** Assume that \( M \) is a microlinear space. If \( \tau \) is a parallel transport on \( M \), then the map \( \nabla : M^D \times M^D \to M^{D \times D} \) defined by

\[
\nabla(t_1, t_2)(h_1, h_2) = \tau_{h_1}(t_1, t_2)(h_2)
\]

is an affine connection on \( M \). Conversely, any affine connection on \( M \) determines a parallel transport \( \tau \) defined by the same formula.

**Proof:** See [3] page 190.

In terms of parallel transport, the notion of a symmetric connection also has a clear geometrical interpretation that the reader is encouraged to find.

When \( M = R^n \), a vector \( t \) at \( a \in R^n \) may be written uniquely as \( t(d) = a + d_b \), by using Kock-Lawvere axiom. The same axiom also implies that a connection can be written as

\[
\nabla(t_1, t_2)(d_1, d_2) = a + b_1 d_1 + b_2 d_2 + \tilde{\nabla}_a(b_1, b_2)d_1d_2
\]

with \( t_i(d) = a + b_i d \) \((i = 1, 2)\).

Thus, \( \nabla \) is completely determined by by

\[
\tilde{\nabla}_a(-, -) : R^n \times R^n \to R^n
\]

The defining properties of \( \nabla \) are translated into properties of \( \tilde{\nabla}_a(-, -) \) and express simply that this last map is bilinear. If \( \nabla \) is symmetric, this map is also symmetric. Notice, however, that in this case such a map is completely determined by its values at the diagonal. Indeed,
Proposition 1.2 Let $\sigma : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a map that is bilinear and symmetric and let $\delta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be defined by $\delta(t) = \sigma(t, t)$. Then

$$\sigma(t_1, t_2) = (1/2)[\delta(t_1 + t_2) - \delta(t_1) - \delta(t_2)]$$

Proof: We have $\delta(t_1 + t_2) = \sigma(t_1 + t_2, t_1 + t_2) = \sigma(t_1, t_1) + 2\sigma(t_1, t_2) + \sigma(t_2, t_2)$, i.e., $\sigma(t_1, t_2) = (1/2)[\delta(t_1 + t_2) - \delta(t_1) - \delta(t_2)]$.

NB For $n = 1$, i.e., for $\sigma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, homogeneity implies symmetry. In fact, by homogeneity

$$\sigma(ax, y) = a\sigma(x, y) = \sigma(x, ay)$$

Taking partial derivatives with respect to $a$ we get

$$x(\partial \sigma / \partial x)(ax, y) = \sigma(x, y) = y(\partial \sigma / \partial y)(x, ay)$$

Taking partial derivatives with respect to $x$ we get

$$x(\partial \sigma / \partial x)(ax, y) + ax(\partial^2 \sigma / \partial^2 x)(ax, y) = (\partial \sigma / \partial x)(x, y).$$

Letting $a = 1$,

$$x(\partial^2 \sigma / \partial^2 x)(x, y) = 0$$

Similarly for $y$. From (1) and (2),

$$\begin{cases}
\sigma(x, 0) = \sigma(0, y) = 0 \\
x(\partial \sigma / \partial x)(x, 0) = 0 \\
y(\partial \sigma / \partial y)(0, y) = 0
\end{cases}$$

Lavendhomme principle implies

$$\begin{cases}
(\partial^2 \sigma / \partial x^2)(x, y) = 0 \\
(\partial \sigma / \partial x)(x, 0) = 0 \\
(\partial \sigma / \partial y)(0, y) = 0
\end{cases}$$

By Taylor developing around $(0, 0)$,

$$\sigma(x, y) = cxy$$

where $c = (\partial^2 \sigma / \partial x \partial y)(0, 0)$. 

4
Thus, $\sigma$ is symmetric, $\delta(x) = cx^2$ is a quadratic form and $\sigma(x, y) = 1/2[\delta(x + y) - \delta(x) - \delta(y)]$.

For $n > 1$ homogeneity does not imply symmetry. In fact, already for $n = 2$ the map $f : R^2 \times R^2 \to R^2$ defined by $f((x_1, x_2), (y_1, y_2)) = (x_1 y_2, x_2 y_1)$ is homogeneous, but not symmetric.

In the following we prove a generalization of the last proposition.

Let $M$ be a microlinear space. A spray on $M$ is a map $\sigma : M^D \to M^{D^2}$ satisfying the conditions

\begin{align*}
(1) & \quad \sigma(t)(d) = t(d) \quad \text{for all } d \in D \\
(2) & \quad \sigma(\lambda t)(\delta) = \sigma(t)(\lambda \delta) \quad \text{for all } \delta \in D_2, \lambda \in R
\end{align*}

For $M = R^n$, we easily see that a spray has the following description

$$\sigma(t)(\delta) = a + b\delta + \tilde{\sigma}_a(b)\delta^2$$

with the homogeneity condition (2)

$$\tilde{\sigma}_a(\lambda b) = \lambda^2 \tilde{\sigma}_a(b)$$

**Proposition 1.3** Given any affine connection (not necessarily symmetric) $\nabla$ on $M$ there is a unique spray $\sigma$ on $M$ such that

$$\sigma(t)(d_1 + d_2) = \nabla(t, t)(d_1, d_2)$$

**Proof:** This follows from the fact that

$$D \xrightarrow{j_1} D \times D \xrightarrow{+} D_2$$

is an $R$-coequalizer, where $j_1(d) = (d, 0)$ and $j_2(d) = (0, d)$. Indeed, for $t \in M^D$, $\nabla(t, t)(d_1, d_2)$ satisfies $\nabla(t, t)(d, 0) = t(d) = \nabla(t, t)(0, d)$. Using the previous $R$-coequalizer, we conclude that there is a unique map $\sigma(t) : D_2 \to M$ satisfying

$$\sigma(t)(d_1 + d_2) = \nabla(t, t)(d_1, d_2)$$

It is easy to check that $\sigma$ satisfies the conditions to be a spray.

The main result of this section is the following version of the Ambrose-Palais-Singer theorem:
Theorem 1.4 Let $M$ be a microlinear space. Then there is a natural one-to-one correspondence between symmetric affine connections $\nabla$ on $M$ and sprays on $M$ given by

$$\sigma(t)(d_1 + d_2) = \nabla(t, t)(d_1, d_2)$$


Before starting the proof, we observe that the following are $R$-coequalizers

(i) $\xymatrix{D & D \times D \ar[r]^+ & D_2}$

(ii) $\xymatrix{D(2) \times D(2) & D(2) \times D(2) \ar[r]^+ & D_2(2)}$

(iii) $\xymatrix{D \times D \times D^{(p_1,mcop23)} & D \times D \ar[r]^m & D}$

(iv) $\xymatrix{D \times D \times D^{(p_1,mcop23)} & D \times D(2) \ar[r]^m & D(2)}$

(v) $\xymatrix{D_2 \times D_2 \times D_2^{(p_1,mcop23)} & D_2 \times D_2(2) \ar[r]^m & D_2(2)}$

where $m$ stands for "multiplication."

From connections to sprays: already done using the $R$-coequalizer (i) (cf. last proposition)

From sprays to connection: we use the $R$-coequalizer (v). Let $\sigma$ be a spray, $t_1, t_2 \in M^D$ such that $t_1(0) = t_2(0) = x$. The map $\tau : D_2 \times D_2(2) \longrightarrow M$ defined by $\tau(\delta, \lambda_1, \lambda_2) = \sigma(\lambda_1 t_1 + \lambda_2 t_2)(\delta)$ equalizes the parallel arrows in (v), by homogeneity of $\sigma$. Therefore, there is a unique map $\nabla'(t_1, t_2) : D_2(2) \longrightarrow M$ such that $\nabla'(t_1, t_2)(\delta \lambda_1, \delta \lambda_2) = \tau(\delta, \lambda_1, \lambda_2) = \sigma(\lambda_1 t_1 + \lambda_2 t_2)(\delta)$. Then $\nabla = \nabla'|_{D \times D} : D \times D \longrightarrow M$ is an affine symmetric connection, as can be easily checked. It remains to prove that these operations are inverse of each other: see [3].
2 Vector bundles and vector fields

Let $M$ be a microlinear space. A vector bundle over $M$ is a map $p : E \to M$ with $E$ microlinear such that every fiber $E_m = p^{-1}(m)$ is equipped with an $R$-module structure satisfying the Kock-Lawvere axiom, i.e., the canonical map

$$H_m : E_m \times E_m \to (E_m)^D$$

defined by $H_m(u, v)(d) = u + dv$ is a bijection.

Examples:

(1) the canonical map $\pi : M^D \to M$ sending a vector $v$ into $v(0)$ is a vector bundle over $M$. (This was already proved. Cf. Proposition 0.1)

(2) the canonical map $\pi_E : E^D \to E$. We denote the $R$-module structure by $\oplus$ and $\odot$ (same as (1))

(3) the map $p^D : E^D \to M^D$. This is not so obvious. We check that its fibers satisfy the Kock-Lawvere axiom. Indeed, given $\phi : D \to (E^D)_t = (p^D)^{-1}(t)$ with $t \in M^D$, consider $\phi_d : D \to E_{t(d)} = p^{-1}(t(d))$ defined by $\phi_d(d') = \phi(d')(d)$. Since $E$ is a vector bundle, there is a unique couple $(u_d, v_d) \in E^2_{t(d)}$ such that $\phi(d') = u_d + d'v_d$ (for $d' \in D$, where $u = [d \mapsto u_d]$ and $v = [d \mapsto v_d]$ are in $(E^D)_t$. The $R$-module structure is defined pointwise and denoted by $+ \text{ and } \odot$ .

(4) the first projection $p_1 : E \times M E \to E$. The $R$-module structure, (again denoted by $\oplus$ and $\odot$) is defined by

$$(u, v) \oplus (u', v') = (u, v + v') \quad \alpha \odot (u, v) = (u, \alpha v)$$

(5) the canonical map $E \times M E \to M$. The $R$-module structure is defined by

$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2) \quad \alpha(u, v) = (\alpha u, \alpha v)$$

(If all $E_m$’s satisfy the Kock-Lawvere axiom, so do all products $E_m \times E_m$.)

(6) the second projection $p_2 : M^D \times M E \to E$. The corresponding $R$-structure is given by

$$(t, v) \oplus (t', v) = (t \oplus t', v) \quad \alpha \odot (t, v) = (\alpha t, v)$$
(7) the first projection \( p_1 : M^D \times_M E \to M^D \). The corresponding structure is defined by

\[(t, v) + (t, v') = (t, v + v') \quad \alpha(t, v) = (t, \alpha v)\]

Whenever \( p : E \to M \) is a vector bundle, we have the following diagram

\[
\begin{array}{ccc}
E \times_M E & \xrightarrow{H} & E^D & \xrightarrow{K} & M^D \times_M E \\
p_1 \downarrow & & \downarrow \pi_E & & \downarrow p_2 \\
E & & & & E
\end{array}
\]

where \( H(u, v)(d) = u + dv \) and \( K(t) = (p \circ t, t(0)) \).

**Proposition 2.1**

1. \( H \) is linear as a map of vector bundles (4) and (2)

2. \( K \) is linear as a map of vector bundles (2) and (6)

3. The sequence of linear maps over \( E \)

\[
0 \to E \times_M E \xrightarrow{H} E^D \xrightarrow{K} M^D \times_M E
\]

is exact, i.e., \( H \) is 1-1 and \( \text{Im}(H) = \text{Ker}(K) \).

**Proof:** (1) and (2). Linearity follows from homogeneity, but this is obvious in each case. (3) \( H \) is monic is a consequence of the fact that the fibers of \( p : E \to M \) satisfy the Kock-Lawvere axiom. It remains to show that \( \text{Im}(H) = \text{Ker}(K) \). Let \( m \in M \) and \( E_m = p^{-1}(m) \) the fiber of \( p \) over \( m \). Then

\[
t \in E^D_m \iff \forall d \in D \ t(d) \in E_m \\
\iff \forall d \in D \ p(t(d)) = m \\
\iff p \circ t = 0_m \\
\iff t \in \text{Ker}(K)
\]

But, by definition of vector bundle, \( t \in E^D_m \) iff there is a unique couple \((u, v) \in E^2_m \) such that \( \forall d \in D \ t(d) = u + dv \), i.e., if \( t = H(u, v) \). This finishes the proof.
A vector \( t \in E^D \) such that \( K(t) = 0 \) is called a \textit{vertical vector}.

Our aim is to generalize the definition of affine connections to vector bundles, and introduce the connection maps.

We recall some notions to be used in the sequel (cf. [3] and [2]).

Let \( M \) be a micro-linear space and let \( p : E \longrightarrow M \) be a vector bundle. An \textit{affine connection} on \( p : E \longrightarrow M \) (briefly, on \( E \)) is a section of \( K \)

\[ \nabla : M^D \times_M E \longrightarrow E^D \]

which is linear with respect to both vector bundles structures \( \oplus \) over \( E \) and \( + \) over \( M^D \). Recall that

\[ \begin{align*}
(u, v) \oplus (u', v') &= (u, v + v'), \\
\alpha \odot (u, v) &= (u, \alpha v) \\
(u_1, v_1) + (u_2, v_2) &= (u_1 + u_2, v_1 + v_2), \\
\alpha \cdot (u, v) &= (\alpha u, \alpha v)
\end{align*} \]

In other words, \( \nabla \) should satisfy the following identities

\[ \begin{align*}
p \circ \nabla(t, v)(d) &= t(d), \quad \nabla(t, v)(0) = v \\
\nabla(\alpha t, v)(d) &= (\alpha \odot \nabla(t, v))(d) = \nabla(t, v)(\alpha d) \\
\nabla(t, \alpha v)(d) &= (\alpha \cdot \nabla(t, v))(d) = \alpha \cdot (\nabla(t, v)(d))
\end{align*} \]

Let \( \nabla : M^D \times_M E \longrightarrow E^D \) be an affine connection on a vector bundle \( p : E \longrightarrow M \). The connection \( \nabla \) induces a split-exact sequence of vector bundles over \( E \) (i.e. with respect to \( \oplus \)):

\[ 0 \longrightarrow E \times_M E \xrightarrow{H} E^D \xrightarrow{K} M^D \times_M E \longrightarrow 0 \]

Hence \( H \) splits as well. Indeed, there is a map \( C_1 : E^D \longrightarrow E \times_M E \) such that

\[ \begin{align*}
C_1 \circ H &= id_{E \times_M E} \\
H \circ C_1 &= id_{E^D} \oplus \nabla K
\end{align*} \]

Indeed, \( K(t \odot \nabla K(t)) = K(t) - K(t) = 0 \). Thus, \( t \odot \nabla K(t) = H(u, v) \) for a unique couple \( (u, v) \in E^2 \). Define \( C_1(t) = (u, v) \). Then \( C_1 \circ H = id_{E \times_M E} \) and \( H \circ C_1 = id_{E^D} \oplus \nabla K \).

We define the connection map associated to \( \nabla \) to be

\[ C : E^D \longrightarrow E \]
given by \( C = p_2 \circ C_1 \).

The connection map \( C : E^D \longrightarrow E \) is bilinear in the sense that for every \((t, t') \in E^D \times_{M^D} E^D\), we have

\[
\begin{cases}
    C(t + t') = C(t) + C(t') \\
    C(t \oplus t') = C(t) + C(t')
\end{cases}
\]

(The two +’s on the right denote addition in \( E_{p(t)} = E_{p(t')}(0) \)).

Intuitively, \( C \) measures the extent to which a tangent vector \( t \in E^D \) coincides with the parallel transport of \( t(0) \) along \( p \circ t \) given by \( \nabla \). (Cf. [3] V3.9-11 and V4.7)

We shall apply these notions to the vector bundle \( E = M^D \) (and \( p = \pi_0 \)). So, we assume that \( M = E^D \). If \( X, Y \) are vector fields, we define the new vector field

\[
\nabla_X Y = C \circ (X \star Y)
\]

with \((X \star Y)(d_1, d_2) = Y_{d_2} \circ X_{d_1}\). We notice that \( X \star Y \) is a transpose of the map \( Y^D \circ X : M \longrightarrow (M^D)^D \). Finally, we define the curvature of the connection \( \nabla \) to be the map

\[
R_{XY} Z = (C \circ C^D - C \circ C^D \circ \Sigma) \circ (Z \star Y \star X)
\]

where

\[
\begin{cases}
    (Z \star Y \star X)(d_1, d_2, d_3) = Z_{d_3} \circ Y_{d_2} \circ X_{d_1} \\
    \Sigma(Z \star Y \star X)(d_1, d_2, d_3) = Z_{d_3} \circ Y_{d_1} \circ X_{d_2}
\end{cases}
\]

Notice that \( Z \star Y \star X \) is a transpose of \( (Z^D)^D \circ Y^D \circ X : M \longrightarrow ((M^D)^D)^D \).

There is a geometrical interpretation of \( \nabla_X Y \):

**Proposition 2.2** Let \( E \longrightarrow M \) be a vector bundle, \( \nabla \) an affine connection on \( E \), \( X \) a vector field on \( M \) and \( Y \) an \( E \)-vector field on \( M \). Then \( \nabla_X Y \) is uniquely determined by the following identity

\[
\forall h \in D \ h(\nabla_X Y)_m = \nabla(d \mapsto \tilde{X}_m(h + d), Y^D \circ X(h))(-h) - Y_m
\]

where \( \tilde{X} : M \longrightarrow M^{D2} \) is the (unique) extension of \( X \).
Proof: (Almost verbatim from [3]). We first notice that $H$ is a bijection (since $E_m$ satisfies the Kock-Lawvere axiom) and this implies that

$$H \circ C_1(t)(h) = p_1 \circ H^{-1} H \circ C_1(t) + h p_2 H^{-1} H C_1(t) = H \circ C_1(t)(0) + h C(t)$$

In particular, for $t = Y^D \circ X - \nabla(X,Y)$ we have

$$\forall h \in D [(Y^D \circ X)_m - \nabla(X_m,Y_m)](h) = Y_m + h(\nabla_X Y)_m$$

The rest of the proof is straightforward and may be found in [3].

2.1 Vector fields over curves

Let $M$ be a microlinear space and $\gamma : R \rightarrow M$. A vector space over $\gamma$ is a map $X : R \rightarrow M^D$ such that the diagram

$$\begin{array}{ccc}
R & \xrightarrow{\gamma} & M \\
 \downarrow & \searrow & \downarrow \pi_M \\
M^D & \xrightarrow{X} & M \\
\end{array}$$

is commutative.

Example: the "velocity field" of $\gamma$, $\gamma^* : R \rightarrow M^D$ defined by $\gamma^*(t)(d) = \gamma(t + d)$.

If $\nabla$ is an affine connection on $M$ and $X$ a vector field over $\gamma$, we define $\nabla_{\gamma^*} X$, a new vector field over $\gamma$, by the equation, for all $h \in D$,

$$h \nabla_{\gamma^*} X(t) = \nabla(\gamma^*(t), X(\gamma(t + h))(-h) - X(\gamma(t))$$

(Notice that existence and uniqueness of $\nabla_{\gamma^*} X(t)$ follow from the validity of Kock-Lawvere axiom for the fiber $M_{\gamma(t)}$.) For the properties of this operation, see [5].
2.2 Computations for $M = \mathbb{R}^n$

Proposition 2.3 (e.g. Fock page 128) If $X = \sum a^i \partial/\partial x^i$ and $Y = \sum b^i \partial/\partial x^i$ are two vector fields on $M = \mathbb{R}^n$, then

$$\nabla_X Y = \sum_{ij} a^i (\partial b^j / \partial x^i) \partial/\partial x^j + \sum_{ijk} \Gamma^k_{ij} a^i b^j \partial/\partial x^k$$

In particular the $k^{th}$-component of $L_X Y$ is

$$(\nabla_X Y)^k = \sum_i a^i (\partial b^k / \partial x^i) + \sum_{ij} \Gamma^k_{ij} a^i b^j$$

Proof: We first prove the particular case $X = \partial/\partial x^i$:

$$\nabla_{\partial/\partial x^i} Y = \nabla_{\partial/\partial x^i} \sum_j b^j \partial/\partial x^j = \sum_j \nabla_{\partial/\partial x^i} b^j \partial/\partial x^j = \sum_j (\partial b^j / \partial x^i) \partial/\partial x^j + b^j \nabla_{\partial/\partial x^i} \partial/\partial x^j$$

$$= \sum_j (\partial b^j / \partial x^i) \partial/\partial x^j + b^j \sum_k \Gamma^k_{ij} \partial/\partial x^k$$

The general case follows from the properties of $\nabla_X Y$ (cf. Lavendhomme [2] page 147). In fact,

$$\nabla_X Y = \nabla \sum a^i \partial/\partial x^i Y$$

$$= \sum_i a^i \nabla_{\partial/\partial x^i} Y$$

$$= \sum_i a^i \nabla_{\partial/\partial x^i} Y$$

$$= \sum_i \sum_j (\partial b^j / \partial x^i) \partial/\partial x^j + b^j \sum_k \Gamma^k_{ij} \partial/\partial x^k$$

3 Geodesics

Proposition 3.1 Let $\gamma : \mathbb{R} \longrightarrow M$ and $\nabla$ a connection on $M$. Then the following are equivalent:

1. $\nabla (\gamma^*, \gamma^*) = \gamma^{**}$

2. $\nabla_{\dot{\gamma}} \gamma^* = 0$

Proof: Unwinding definitions and writing everything explicitly, these expressions are shorthands for
1. \( \gamma^\bullet(t + h_1) = \tau_{h_1}(\gamma^\bullet(t), \gamma^\bullet(t)) \)

2. \( \tau_{-h}(\gamma^\bullet(t + h), \gamma^\bullet(t + h)) = \gamma^\bullet(t) \)

We obtain the second from the first by substituting
\[
\begin{align*}
  h_1 & \mapsto -h \\
  t & \mapsto t + h
\end{align*}
\]

To obtain the other direction, we make the substitution
\[
\begin{align*}
  h & \mapsto -h_1 \\
  t & \mapsto t + h_1
\end{align*}
\]

We say that \( \gamma : R \rightarrow M \) is a geodesic iff \( \gamma \) satisfies any of the conditions of the proposition.

Thus, geodesic curves have a clear geometrical content: “by parallel transporting an infinitesimal portion of the curve (seen as a tangent vector) along itself, the result is again an infinitesimal portion of the curve, i.e., a new tangent vector to the curve.” No metrics are needed to define this notion.

Let \( M \) is an n-dimensional manifold. By introducing \( x \)-coordinates, we may assume that \( M = R^n \), and use Tensor Calculus (“les débauches d’indices” according to Élie Cartan). In this case,
\[
\nabla : M \times R^n \rightarrow M \times R^n \times R^n
\]

is completely determined by by its last component \( \nabla_4 \), since \( K \) is a projection and we define the Christoffel symbols of the second kind by
\[
(\Gamma^i_{ji}(p)) = -\nabla_4(p, e_i, e_j)(e_l)
\]

We shall usually omit ‘x’ from the notation, if the coordinate frame is clear from the context.

**Lemma 3.2** If \( \nabla \) is a connection, \( \nabla^4 \) is bilinear.

**Proof:** Immediate from the properties \( \nabla(\lambda b_1, b_2)(d_1, d_2) = \nabla(b_1, b_2)(\lambda d_1, d_2) \) and \( \nabla(b_1, \lambda b_2)(d_1, d_2) = \nabla(b_1, b_2)(d_1, \lambda d_2) \).
Proposition 3.3 A geodesic \( \gamma = (x^1, \ldots, x^n) \) satisfies the second order differential equation

\[
d^2 x^k / dt^2 + \sum_{i,j=1}^n \Gamma^k_{\alpha \beta} (dx^\alpha / dt)(dx^\beta / dt) = 0
\]

Proof: Recall that the vector field along \( \gamma \), \( \gamma^\bullet : \mathbb{R} \rightarrow M^D \) is defined by \( \gamma^\bullet(t) = [d \mapsto \gamma(t + d)] \), i.e., \( \gamma^\bullet(t)(d) = \gamma(t + d) \). It follows that \( \gamma^{\bullet\bullet} : \mathbb{R} \rightarrow (M^D)^D \) is given by \( \gamma^{\bullet\bullet}(t)(d_1, d_2) = \gamma(t + d_1 + d_2) \). Hence

\[
\gamma^\bullet(t)(d) = (x^1(t + d), \ldots, x^n(t + d))
\]

\[
= (x^1(t) + d(dx^1 / dt)(t), \ldots, x^n(t) + d(dx^n / dt)(t))
\]

\[
= \bar{x}(t) + d(dx / dt)(t)
\]

with \( \bar{x} = (x^1(t), \ldots, x^n(t)) \) and \( dx / dt(t) = (dx^1 / dt(t), \ldots, dx^n / dt(t)) \). Leaving the "t" out,

\[
\gamma^{\bullet\bullet}(d_1, d_2) = (x^1 + d_1(dx^1 / dt) + d_2(dx^1 / dt), \ldots, x^n + d_1(dx^n / dt) + d_2(dx^n / dt) + d_1 d_2 (d^2 x / dt^2))
\]

\[
= \bar{x} + d_1(dx / dt) + d_2(dx / dt) + d_1 d_2 (d^2 x / dt^2)
\]

Similarly,

\[
\nabla(\gamma^\bullet, \gamma^\bullet)(d_1, d_2) = \bar{x} + d_1(dx / dt) + d_2(dx / dt) + d_1 d_2 \nabla^4_{dx / dt, dx / dt}
\]

The equation of the geodesic becomes then

\[
(*) \quad d^2 \bar{x} / dt^2 = \nabla^4_{dx / dt, dx / dt}
\]

Writing \( dx / dt = \sum_\alpha (dx^\alpha / dt) \partial / \partial x^\alpha \) and using the bi-linearity of \( \nabla^4 \) we have

\[
\nabla^4_{dx / dt, dx / dt} = \sum_{\alpha \beta} (dx^\alpha / dt)(dx^\beta / dt) \nabla^4(\partial / \partial x^\alpha, \partial / \partial x^\beta)
\]

Taking the \( k^{th} \) component in the equation \( (*) \), we finally obtain

\[
d^2 x^k / dt^2 + \sum_{\alpha \beta} \Gamma^k_{\alpha \beta}(dx^\alpha / dt)(dx^\beta / dt) = 0
\]

This is the expression found in texts of Differential Geometry (See e.g. [6]).

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Proposition 3.4 Given an affine connection $\nabla$ on $M$, there is a symmetric affine connection $\tilde{\nabla}$ having the same geodesics.

Proof: Let $\nabla$ be an affine connection and let $\sigma : M^D \rightarrow M^{D_2}$ be the unique spray associated to $\nabla$, i.e., defined by the formula $\sigma(t)(d_1 + d_2) = \nabla(t, t)(d_1, d_2)$. Define $\tilde{\nabla}$ to be the symmetric affine connection associated with $\sigma$ (given by the Ambrose-Palais-Singer Theorem). We claim that $\nabla$ and $\tilde{\nabla}$ have the same geodesics. In fact, if $\gamma : \mathbb{R} \rightarrow M$ is a curve,

$\tilde{\nabla}(\gamma^\ast(t), \gamma^\ast(t))(d_1, d_2) = \sigma(\gamma^\ast(t)(d_1 + d_2)) = \nabla(\gamma^\ast(t), \gamma^\ast(t))(d_1, d_2)$

(the first by definition of $\tilde{\nabla}$ (cf. Ambrose-Palais-Singer theorem already mentioned); the second by definition of $\sigma$). But $\gamma$ is a geodesic for a connection $\nabla$ iff $\nabla(\gamma^\ast(t), \gamma^\ast(t)) = \gamma^{\ast\ast}(t)$. Thus, $\nabla$ and $\tilde{\nabla}$ have the same geodesics.

This shows that, so far as geodesics of a connection are concerned, we may assume, without loss of generality, that the connection is symmetric.

4 Transformation law for vectors and vector fields

In this section we study how vectors transform from $x$-coordinates to $x'$-coordinates.

Let $\phi = (\phi^1, \ldots, \phi^n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a diffeomorphism and $v = (v^1, \ldots, v^n) : D \rightarrow \mathbb{R}^n$ a tangent vector. We want to find the the law of transformation of the components of $v$ under $\phi$. Assume that $v^i(d) = a^i + d b^i$ ($i = 1, \ldots, n$). We have

$$D \xrightarrow{v} R^n \xrightarrow{\phi} R^n$$

$$(\phi \circ v)(d) = (\phi^1(a) + d \sum_j b_j (\partial \phi^1 / \partial x^j)(a), \ldots, \phi^n(a) + d \sum_j b_j (\partial \phi^n / \partial x^j)(a))$$

This means that the $i^{th}$-component of $\phi \circ v$ is

$$(\phi \circ v)^i = \sum_j b^j (\partial \phi^i / \partial x^j).$$
Define $v^i = \pi_i(\phi \circ v)$. By identifying $v^i$ with $b^i$ and letting $x'^i(x) = \phi^i(x)$, as is common in texts of GR, our previous expression can be written

$$v^i = \sum_j v^j \left( \frac{\partial x'^i}{\partial x^j} \right)$$

This is the transformation rule "from the $x$ variables to the $x'$ variables". Using a terminology used extensively, vectors transform "contravariantly." Clearly this argument gives the law of transformation for vector fields by just interpreting $v'^i$ and $v^j$ as components of a vector field.

There is a corresponding transformation law for co-vectors which can be deduced from this one, namely

$$\frac{\partial}{\partial x^\mu} = \sum_\alpha (\partial \xi^\alpha / \partial x^\mu) \frac{\partial}{\partial \xi^\alpha}$$

### 4.1 Transformation law for $\Gamma^k_{ij}$

In this section, we present the proof of Synge/Schild [7] of the transformation rule for $\Gamma^k_{ij}$ when we pass from $x$-coordinates to $x'$-coordinates:

$$(\Gamma^\gamma_{\alpha\beta})_{x'} = \sum_{ijk} (\Gamma^k_{ij}) + x(\partial x^\gamma / \partial x^k)(\partial x^i / \partial x^\alpha)(\partial x^j / \partial x^\beta) + \sum_k (\partial^2 x^k / \partial x^\alpha \partial x^\beta)(\partial x^\gamma / \partial x^k)$$


Start with a curve $\gamma : R \rightarrow M$. Then we know (previous section) that $V = \gamma^\bullet + \nabla(\gamma^\bullet, \gamma^\bullet)$ is a contravariant vector field. In $x$ coordinates, the $\gamma$-th component of $V$ is

$$v^\gamma = \frac{d^2 x^\gamma}{dt^2} + \sum_{\alpha\beta} \Gamma^\gamma_{\alpha\beta}(dx^\alpha / du)(dx^\beta / du)$$

Thus, by the transformation law of contravariant vector fields,

$$v'^\gamma = \sum_k v^k \left( \frac{\partial x'^\gamma}{\partial x^k} \right)$$

**Lemma 4.1 (2.504)**

$$(d^2 x^k / dt^2) \left( \frac{\partial x'^\gamma}{\partial x^k} \right) = (d^2 x^\gamma / dt^2) + \sum_{\alpha\beta} \left( \frac{\partial x'^\gamma}{\partial x^k} \left( \frac{\partial^2 x^\gamma}{\partial x^\alpha \partial x^\beta} \right) dx^\gamma / dt \right) \left( dx^\beta / dt \right)$$
Proof: By Leibniz’s rule,
\[
d^2x^k/dt^2 = d/dt[(\sum_{\alpha} \partial x^k/\partial x^\alpha)(dx^\alpha/dt)] = \sum_{\alpha}(\partial x^k/\partial x^\alpha)(d^2x^\alpha/dt^2) + \sum_{\alpha\beta}(\partial^2 x^k/\partial x^\alpha\partial x^\beta)(dx^\alpha/dt)(dx^\beta/dt)
\]
Multiplying both sides by \(\partial x^\gamma/\partial x^k\) we obtain the result sought.

Corollary 4.2 (Transformation law for \(\Gamma\))
\[
(\Gamma_{\alpha\beta}^\gamma) x = \sum_{ijk}(\Gamma_{ij}^k)x(\partial x^\gamma/\partial x^k)(\partial x^i/\partial x^\alpha)(\partial x^j/\partial x^\beta)
+ \sum_k(\partial^2 x^k/\partial x^\alpha\partial x^\beta)(\partial x^\gamma/\partial x^k)
\]
Proof: From (2.502)
\[
d^2x^\gamma/dt^2 + (\Gamma_{\alpha\beta}^\gamma)x(dx^\alpha/dt)(dx^\beta/dt) = \sum_k(d^2x^k/dt^2 + (\Gamma_{\alpha\beta}^k)x(dx^\alpha/dt)(dx^\beta/dt))(\partial x^\gamma/\partial x^k)
\]
Using 2.504, changing bound variables \(\alpha, \beta\) by \(i, j\) and noticing that
\[
\begin{align*}
dx^i/dt &= \sum_{\alpha}(\partial x^i/\partial x^\alpha)(dx^\alpha/dt) \\
dx^j/dt &= \sum_{\alpha}(\partial x^j/\partial x^\alpha)(dx^\alpha/dt)
\end{align*}
\]
we obtain
\[
A_{\alpha\beta}^\gamma(dx^\alpha/dt)(dx^\beta/dt) = 0
\]
with
\[
A_{\alpha\beta}^\gamma = (\Gamma_{\alpha\beta}^\gamma)x - \sum_{ijk}(\Gamma_{ij}^k)x(\partial x^\gamma/\partial x^k)(\partial x^i/\partial x^\alpha)(\partial x^j/\partial x^\beta)
- \sum_k(\partial x^\gamma/\partial x^k)(\partial^2 x^k/\partial x^\alpha\partial x^\beta)
\]
Now, \(A_{\alpha\beta}^\gamma\) is a function on \(M\), independent of both \(dx^\alpha/dt\) and \(dx^\beta/dt\).
Therefore, \(A_{\alpha\beta}^\gamma = 0\). In more details, let \(p \in M\). Let \(\gamma\) be a geodesic starting at \(p\) with initial velocity \((1, \ldots, 1)\) in the \(x^i\)-coordinates. Then \(A_{\alpha\beta}^\gamma(p) = 0\).
We have obtained the transformation law.

4.2 Transformation law for \(\Gamma\) (bis)
Recall from [5] (page 4) that
\[
(*) \left(\nabla_{\partial/\partial x^i}\partial/\partial x^j\right)_x = \sum_{k=1}^n \Gamma_{ij}^k(x)\partial/\partial x^k|_x
\]
By the properties of $\nabla$, if $X = \sum_i a^i \partial / \partial x^j$ and $Y = \sum_j b^j \partial / \partial x^i$, then ([5] page 5)

$$(* *) \nabla X Y = \sum_{ij} a^i (\partial b^j / \partial x^i) \partial / \partial x^j + \sum_{ijk} \Gamma^k_{ij} a^i b^j \partial / \partial x^k$$

In $x'$ coordinates, (*) becomes

$$(*) (\nabla \partial / \partial x^\alpha \partial / \partial x^\beta)_{x'} = \sum_\gamma \Gamma^\gamma_{\alpha\beta}(x') \partial / \partial x'^\gamma |_{x'}$$

and the $\Gamma$'s are determined by this equation, since $\{\partial / \partial x'^\gamma \}_{x'}$ is a basis.

For the particular case

$$X = \partial / \partial x^\alpha = \sum_i (\partial x^i / \partial x^\alpha) \partial / \partial x^i$$
$$Y = \partial / \partial x^\beta = \sum_j (\partial x^j / \partial x^\beta) \partial / \partial x^j$$

$a^i = \partial x^i / \partial x^\alpha$ and $b^j = \partial x^j / \partial x^\beta$ and

$$(* *) \nabla \partial / \partial x^\alpha \partial / \partial x^\beta = \sum_{ij} (\partial x^i / \partial x^\alpha) (\partial / \partial x^j) + \sum_{ijk} \Gamma^k_{ij} (\partial x^i / \partial x^\alpha) (\partial x^j / \partial x^\beta) \partial / \partial x^k$$

From

$$\partial / \partial x^k = \sum_\sigma (\partial x^\sigma / \partial x^k) \partial / \partial x^\sigma$$

we conclude that the $\gamma$th term of the second summand of $(* *)$ is

$$\sum_{ijk} \Gamma^k_{ij} (\partial x^i / \partial x^\alpha) (\partial x^j / \partial x^\beta) (\partial x^\gamma / \partial x^k)$$

Similarly, from

$$\partial / \partial x^j = \sum_\sigma (\partial x^\sigma / \partial x^j) \partial / \partial x^\sigma$$

we conclude that the $\gamma$th term of the first summand is

$$\sum_{ij} (\partial x^i / \partial x^\alpha) (\partial / \partial x^j) (\partial x^\gamma / \partial x^j)$$
$$= \sum_{ij} (\partial x^i / \partial x^\alpha) \sum_\sigma (\partial x^\sigma / \partial x^i) \partial / \partial x^\sigma (\partial x^j / \partial x^\beta) (\partial x^\gamma / \partial x^j)$$
$$= \sum_{i\sigma} (\partial x^i / \partial x^\sigma) (\partial x^\sigma / \partial x^i) (\partial x^\gamma / \partial x^j) (\partial x^\gamma / \partial x^j)$$
$$= \sum_{i\sigma} \delta^\sigma_{i\sigma} (\partial^2 x^i / \partial x^\alpha \partial x^\beta) (\partial x^\gamma / \partial x^j)$$
$$= \sum_{ij} (\partial^2 x^i / \partial x^\alpha \partial x^\beta) (\partial x^\gamma / \partial x^j)$$

This gives the law of transformation

$$(\Gamma^\gamma_{\alpha\beta})_{x'} = \sum_{ijk} (\Gamma^k_{ij})_x (\partial x^i / \partial x^\alpha) (\partial x^j / \partial x^\beta) (\partial x^\gamma / \partial x^k) + \sum_i (\partial^2 x^i / \partial x^\alpha \partial x^\beta) (\partial x^\gamma / \partial x^i)$$
4.3 A note on the equivalence between definitions of tangent vectors in classical DF and SDG

Recall that classically, a tangent vector at \( p \) is defined as a linear derivation at \( p \)

\[ V_p : C^\infty(U) \rightarrow \mathbb{R} \]

where \( p \in U \) and \( U \) is an open subset of \( M \). (NB: this definition is independent of \( U \) CHECK!)

On the other hand, a tangent vector at \( p \) in SDG is simply a map \( v_p : D \rightarrow M \) with \( v_p(0) = p \).

**Proposition 4.3** If \( M \) is a smooth \( n \)-dimensional manifold, there is a one-to-one correspondence between

Maps \( v : D \rightarrow M \) with \( v(0) = p \)

**Proof:** We may assume as well that \( M = \mathbb{R}^n \) and \( p = 0 \). This proposition is a consequence of Hadamard’s. Let \( V_p \) be a \( \mathbb{R} \)-linear derivation at \( p \) and \( f : U \rightarrow \mathbb{R} \) with \( 0 \in U \) and \( U \) convex open in \( M \). By Hadamard’s

\[ f(x) = f(0) + \sum_i x^i g_i(x) \]

Since \( V_p(e) = 0 \) (as easily checked), \( V_0(f) = V_0(\sum_i x^i g_i(x)) \). Hence,

\[ V_0(f) = V_0(\sum_i x^i g_i(x)) \]

\[ = \sum_i V_0(x^i g_i(x)) \]

\[ = \sum_i g_i(0)V_0(x^i) + 0 \]

\[ = \sum_i \partial f/\partial x^i(0)V_0(x_i) \]

\[ = (\sum_i a_i \partial/\partial x^i|_0)(f) \]

Thus the operator \( V_0 \) is completely determined by the \( n \)-tuple \((a_1, \ldots, a_n)\) (since the operators \( \partial/\partial x^i|_0 \)’s are clearly linearly independent)

By Kock-Lawvere, a map \( v_0 : D \rightarrow \mathbb{R}^n \) is of the form \( v_0(d) = (a_1d, \ldots, a_nd) \) for unique \( n \)-tuple \((a_1, \ldots, a_n)\). Thus the correspondence can be described as follows: if \( V_0 \) is given, we let \( v_0(d) = \sum_i V_0(x^i)d \). Conversely, if \( v_0 \) is given, say, \( v_0(d) = (a_1d, \ldots, a_nd) \), we let \( V_0 = \sum_i a_i \partial/\partial x^i|_0 \).

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References


