A derivation of Einstein's vacuum field equations

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For Mihály: teacher, student, collaborator, friend

1 Introduction

In his work on General Relativity ([2]), Einstein started from the field formulation of Newton's gravitational theory due to Poisson, i.e., the equation

$$\nabla^2\phi = 4\pi\kappa\rho$$

where ϕ is the gravitational potential, $\nabla^2\phi = \sum_\alpha \partial^2\phi/\partial(x^\alpha)^2$ the Laplacian operator, κ a constant and ρ is the density of matter.

He proceeded to find the analogue of the Laplacian operator in the relativistic 4-dimensional space-time manifold that replaced Newton's 3-dimensional space. For this, he had first to look for the analogue of gravitational potentials. To achieve this, he postulated the existence of a metric that he took as the analogue of the potential. Then he looked for the analogue of the Laplacian operator. He did this somewhat formally: only second order derivatives of the metric should appear, etc. Finally, he sought the analogue of density of matter.

Since we are interested only on the vacuum equations, we start rather from Laplace equation

$$\nabla^2\phi = 0$$

where ϕ is the gravitational potential. Thus, we leave out the question of the analogue of the density of matter.

This limitation has three motivations

1. According to Einstein himself ([4]), "... [the vacuum equations] are the only equations which follow without ambiguity from the fundamental hypothesis of General Relativity...". On the analogue of the density of matter of Poisson's equation equations he says ([5]): " The right side of the [field] equations [the tensor $-kT_{ik}$] is a formal condensation of all things whose comprehension in the sense of a field theory is still problematic." Further, he says "...this formulation was merely a makeshift..."

2. The vacuum field equations suffice to derive the Schwarzschild solution and deduce the testable consequences of the theory that Einstein mentioned in his work, i.e., the gravitational red shift of spectral lines, the deflection of light by the sun and the precession of the perihelia of the orbits of the inner planets.

3. The whole derivation can be carried out from the assumption of a space-time with a symmetric connection, rather than a metric.

Returning to Laplace equation, a little familiarity with Linear Algebra brings to light the trivial observation that $\nabla^2\phi$ is the *trace* of the linear transformation Ψ given by the matrix $(\partial^2\phi/\partial x^\alpha \partial x^\beta)_{\alpha\beta}$. This suggests to concentrate on Ψ instead and find a relativistic analogue of this linear transformation.

This is what Sachs and Wu do in their approach to derive Einstein's field equations in [13]. To achieve this aim, they provide a geometrical interpretation of Ψ in terms of acceleration of a neighbor (a vector field along a curve, to be defined below) whose analogue is readily available in the relativistic theory, namely covariant acceleration of the corresponding relativistic neighbor. The map Ψ is in fact defined from Newton law of motion of freely falling particles (in a gravitational field). The corresponding map in Relativity is likewise defined in terms of the Einstein's Law of Motion of free falling particles.

Einstein's vacuum field equation, in analogy with Laplace vacuum field formulation of Newtonian theory can be formulated as the statement that the trace of the relativistic analogue of Ψ is zero.

Thus, we derive the vacuum field equations from the law of motion in both cases.

Our approach to derive Einstein's vacuum field equations follows that of Sachs and Wu. However, and in spite of the same starting point, our derivation differs considerably from theirs as anyone browsing their book can see. Among other differences, we do not require a metric but the simpler notion of parallel transport, as mentioned before.

Another difference from both Einstein's and Sachs and Wu's approaches is the extensive use of infinitesimals. In fact, we place ourselves in the context of Synthetical Differential Geometry (SDG). SDG is a mathematical theory whose aim is to provide a rigorous foundation for infinitesimals and infinitesimal structures as used by geometers such as Sophus Lie, Elie Cartan as well as some physicists and engineers. (See [7],[10],[11]).

In SDG, the reals R have the structure of a (non-trivial) ring, not a field, and the main axiom states one of the basic insights of the creators of calculus, namely, that "in the infinitely small, every curve is a line in a unique way." To formulate this axiom, let $D = \{x \in R | x^2 = 0\}$. An infinitesimal curve is a function $f \in R^D$.

Axiom of Kock-Lawvere (KL): The map $R \times R \longrightarrow R^D$ that sends (a, b) into the curve $[d \mapsto a + db]$ is a bijection.

As an application, let $f \in R^R$ and $x_0 \in R$. To define $f'(x_0)$, the derivative of f at the point x_0, consider the infinitesimal curve $\gamma : D \longrightarrow R$ defined by $\gamma(d) = f(x_0 + d)$. By the axiom, $\gamma(d) = a + db$ for a unique couple (a, b). We define $f'(x_0) = b =$ the slope of the line. Thus, $f(x_0 + d) = f(x_0) + df'(x_0)$, since clearly $a = f(x_0)$.

The usual properties of derivatives can be proved, by algebraic manipulations but we will not go into these matters. (See books cited above).

An important property of D that holds in practically all models is that the functor $(-)^D$, itself a right adjoint of $(-) \times D$, has a right adjoint $(-)^{1/D}$. Thus $(-) \times D \dashv (-)^D \dashv (-)^{1/D}$. In particular (and this is the only property of D we will use) $(-)^D$ preserves arbitrary unions.

Besides D, the first order infinitesimals, there are higher order ones such as $D_n = \{x \in R | x^{n+1} = 0\}$ and $D_\infty = \{x \in R | \exists n \ x^n = 0\}$, the nilpotent reals. Infinitesimals make the main notions of Differential Geometry such as connexion, parallel transport, covariant derivative, etc., quite intuitive and (several) proofs simple algebraic computations. As an example, the tangent bundle of a space M is simply the exponential space M^D and the structure map $\pi_M : M^D \longrightarrow M$ is just "evaluation at 0," i.e., $\pi_M(\gamma) = \gamma(0)$. For "good" spaces, the so-called *microlinear spaces* that play a role in SDG similar to smooth manifolds in classical Differential Geometry, the fiber $M_m = \pi_M^{-1}(m)$ is an R-module.

Before launching ourselves head on into our main concern, let us notice a circumstance that will simplify enormously our task: the notions of velocity and acceleration of particles in Newtonian theory are defined *at a single point*. So is the Laplacian. These notions, for their definition, require to take derivatives which according to SDG require functions to be defined only on an *infinitesimal neighborhood* of the point (rather that an ordinary neighborhood as in classical DG). From now on everything will be infinitesimal: manifolds, geodesics, vector fields, etc.

In more details, open covers to define manifolds will be replaced by infinitesimal covers to obtain D_∞-manifolds.

A D_∞-*manifold of dimension* n is a space having the property that each point has a neighborhood isomorphic to D_∞^n. Clearly, D_∞-manifolds are closed under product and the dimension of the product is the sum of the dimension of the factors. Furthermore, embedded (classical) manifolds in smooth toposes are D_∞-manifolds. These manifolds are microlinear spaces.

Notice that in D_∞-manifolds we can introduce coordinates and make calculations with them, just as in classical manifolds. Although our definitions are invariant and hence independent of coordinates, these will allow to compare ours with the classical ones in the literature.

Trajectories will be *infinitesimal* curves with domain D_∞.

Furthermore, vector fields on a D_∞ manifold will be maps $Q : U \longrightarrow U^D$ with U an infinitesimal neighborhood of M.

Working with these infinitesimal objects, we are able to carry out our whole derivation completely in the realm of the infinitesimal. This simplifies things considerably. Roughly speaking algebra rather than analysis is required to carry out our proof and in fact, the whole derivation of Einstein's equations from the Sachs and Wu reformulation of Laplace's equation can be carried out synthetically (or axiomatically). As an example, existence and uniqueness of "infinitesimal" solutions of differential equations can be proved (as done e.g. in [8]). Other examples will appear below.

From a classical point of view we need to deal only with "formal analysis" (i.e., analysis with elements of formal power series, rather than ordinary functions) to carry out our derivation.

The reader is referred to [10] and [12] for all that we need from SDG.

2 Acknowledgements

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3 Newton's theory of Gravitation

Newton's law of motion states that the trajectory of particles of freely falling particles in the gravitational field obey the second-order differential equation

$$\gamma''(t) + \nabla(\phi(\gamma(t)) = 0 \quad \text{for all } t \in D_\infty.$$

In other words, the trajectory are solutions of the autonomous second order differential equation G :

$$y'' + \nabla(\phi(y)) = 0, \quad \text{where } \phi \text{ is the gravitational potential.}$$

written in vector form. Equivalently, the system

$$(y^\alpha)'' + \nabla^\alpha(\phi(y^1, y^2, y^3)) = 0 \quad \text{for } \alpha = 1, 2, 3$$

written in terms of scalar components.

In invariant terms, this differential equation may be considered as a symmetric vector field $G : E^D \longrightarrow (E^D)^D \approx E^{D\times D}$ satisfying $s \circ G = G$ where $s : E^{D\times D} \longrightarrow E^{D\times D}$ is the map induced by the twisting map $\tau : D \times D \longrightarrow D \times D$ given by $\tau(d_1, d_2) = (d_2, d_1)$. (Cf. "Second order DE's and geodesics.pdf" in [12]). The solutions of G are the trajectories of freely falling particles.

The fact that G is not an ordinary vector field on E (since it is defined on E^D rather than E) creates problems for further developments.

Fortunately, G is reducible to a family $\{G^u\}_u$ of ordinary vector fields indexed by non zero vectors in the sense given by the following

Proposition 3.1 *Let $\gamma : D_\infty \longrightarrow E$ be a curve with $\gamma(0) = x$ and $\gamma^\bullet(0) = u$, a non zero vector in E_x. Then γ is a solution of G iff γ is an integral curve of G^u.*

Proof: See section 5 (Proposition 5.2)

By defining a *G-vector field* to be a vector field whose integral curves are solutions of G, we have

Corollary 3.2 *Every G^u is a G-vector field.*

The key notion of the Sachs and Wu approach is that of (the trajectories of) a swarm of infinitely close particles freely falling in the gravitational field.

Let $\{\gamma_h\}_{h\in D}$ be the trajectories of the swarm. We may view this family as a single map

$$W : D_\infty \longrightarrow E^D$$

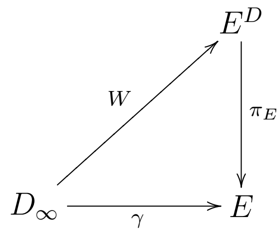
by defining $W(t)(h) = \gamma_h(t)$.

In fact, it is clear that we have the following one-to-one correspondence for every $v \in M_x = \pi_M^{-1}(x)$.

$$\frac{\text{Families } \{\gamma_h : D_\infty \longrightarrow E\}_{h\in D} \text{ with } \gamma_h(0) = v(h) \text{ for all } h \in D}{\text{Maps } W : D_\infty \longrightarrow E^D \text{ with } W(t)(0) = \gamma_0(t) \text{ and } W(0) = v.}$$

Letting $\gamma = \gamma_0$, we have $W(t)(0) = \gamma_0(t) = \gamma(t)$.

Then W is *a vector field along* γ in the sense that the following diagram



is commutative.

Let $\gamma : D_\infty \longrightarrow E$ be a curve and $x = \gamma(0)$, $u = \gamma^\bullet(0)$ a non-zero vector. A *neighbor of* γ *in a vector field* Q is a vector field along γ, $W : D_\infty \longrightarrow E^D$ such that for every $h \in D$, the curve γ_h defined by $\gamma_h(t) = W(t)(h)$ is an integral curve of Q.

Proposition 3.3 *Let* γ *be a trajectory of a free falling particle,* $\gamma(0) = x$, $\gamma^\bullet(0) = u$ *(a non-zero vector) and let* $v \in E_x$*. Then there is a unique neighbor* W^v *of* γ *in* G^u *such that* $W^v(0) = v$.

Proof: Define $W^v(t)(h) = \gamma_h(t)$, where γ_h is the integral curve of G^u with $\gamma_h(0) = v(h)$. Uniqueness is a consequence of uniqueness of solutions of DEs. (cf. [8]) □

By an application of KL, $W^v(t)(h) = \gamma_0(t) + h\delta(t) = \gamma(t) + h\delta(t)$ for a unique function δ.

Let $u \in E_x$ be a non zero vector. By existence and uniqueness of second-order DEs there is a (unique) solution $\gamma : D_\infty \longrightarrow E$ of G with $\gamma(0) = x$ and $\gamma^\bullet(0) = u$. Thus γ is an integral curve of G^u. Define a map

$$\Psi^u : E_x \longrightarrow E_x$$

as follows. Let $v \in E_x$ be arbitrary. Pick the unique neighbor W^v of γ in G^u with $W^v(0) = v$. Define a vector field $(W^v)''$ along γ by

$$(W^v)''(t)(h) = \gamma(t) + h\delta''(t)$$

for every t.

Finally, define

$$\Psi^u(v) = (W^v)''(0).$$

This vector field along γ is already well-known in differential geometry. In fact, leaving out the superscript v from W,

Proposition 3.4 *The following holds*

$$W'' = \nabla^2_{\gamma^\bullet} W$$

where the connection is the Euclidean (or flat) one.

Proof: Recall that $\nabla_{\gamma^\bullet} W$ is uniquely defined by the property $h\nabla_{\gamma^\bullet} W(t) = \nabla(\gamma(t+h), W(t+h))(-h) - W(t)$ for every $h \in D$. In words, we transport $W(t+h)$ back along γ to $\gamma(t)$ and subtract $W(t)$. (Cf. "covariantderivation.pdf" page 7 in [12].)

Since the connection is euclidean, vectors are preserved and the result is just $h\nabla_{\gamma^\bullet} W(t) = \lambda d[\gamma(t) + d\delta(t+h)] - \lambda d[\gamma(t) + d\delta(t)] = \lambda d(\gamma(t) + d\delta'(t))$ with $\delta(t)$ given by $\gamma_d(t) = \gamma(t) + d\delta(t)$ (K/L).

Iterating,

$$(\nabla^2 W)(t) = \lambda d(\gamma(t) + d\delta''(t)).$$

finishing the proof of the proposition.

We now prove the main theorem of this section. Notice first that E_x is canonically isomorphic to R^3 by the isomorphism $v \in E_x \mapsto v_0 \in R^3$ where $v(d) = x + dv_0$ (K/L). In classical notation, $v = (x, v_0)$ with $x = (x^1, x^2, x^3)$ and $v_0 = (v_0^1, v_0^2, v_0^3)$.

Thus, we may view Ψ^u as a map $\Psi^u : R^3 \longrightarrow R^3$ and $\Psi(v) = W''(0) = \delta''(0)$.

Theorem 3.5 (cf. [13]) *The map $\Psi^u : R^3 \longrightarrow R^3$ is an R-linear transformation. In fact $(-\Psi^u)$ is the linear transformation defined by the matrix $(\partial^2\phi/\partial x^\alpha \partial x^\beta)_{\alpha\beta}$. In particular, Ψ^u is independent of u.*

Proof: Since G^u is a G-field, the trajectory γ_h is a solution of G for each $h \in D$, i.e.,

$$(\gamma_h)''(t) = -\nabla\phi(\gamma_h(t)) \quad \text{for each t}$$

where ϕ is the gravitational potential. Thus, at time t,

$$-h\delta''(t) = -[(\gamma_h'') - (\gamma)''](t) = \nabla\phi(\gamma_h(t)) - \nabla\phi(\gamma(t))$$

In particular, at time $t = 0$,

$$-h\delta''(0) = \nabla\phi(\gamma_h(0)) - \nabla\phi(\gamma(0))$$

By developing the RHS in Taylor series, we have that its α^{th} component is

$$h \sum_{\beta} (\partial^2 \phi / \partial x^\alpha \partial x^\beta)(x) v_0^\beta$$

Then $\Phi(v) = -\delta''(0) = -\Psi^u(v)$ where

$$\Phi : R^3 \longrightarrow R^3$$

is the linear transformation defined by the matrix $((\partial^2 \phi / \partial x^\alpha \partial x^\beta)(x))_{\alpha\beta}$. Writing Ψ for Ψ^u, we may formulate

Newton's vacuum field equation

$$trace(\Psi) = 0$$

In fact, this is just the Laplace equation.

NB We may view the preceding as the construction of a map

$$\begin{cases} E_x^* & \longrightarrow & Lin_R(E_x, E_x) \\ u & \mapsto & \Psi^u \end{cases}$$

where E_x^* the set of non zero tangent vectors in E_x, from the law of motion G (the Newtonian law).

Indeed, let $u \in E_x^*$. By existence and uniqueness of solutions of (autonomous) second order equations there is a unique solution $\gamma : D_\infty \longrightarrow E$ of G with $\gamma(0) = x$ and $\gamma^\bullet(0) = u$. Now, proceed as before starting with Proposition 3.3.

The vacuum field equation is just the statement $trace(\Psi^u) = 0$ for every u. In this particular case, Ψ^u is independent of u.

The same scheme will be used to derive the vacuum field equations in Einstein's relativistic theory of gravitation that we now discuss.

4 Einstein's Relativistic Theory of Gravitation

The General Theory of Relativity or rather the Relativistic Theory of Gravitation (as it should be properly called according to the Russian physicist V.Fock [6]) rests on the following fundamental assumptions.

1. Space-time is a 4-dimensional "curved" manifold.

2. Free falling particles describe geodesics ("straightest lines") in this manifold.

3. There is a reciprocal action of matter on curvature and curvature on matter described by Einstein's field equations. In the words of John Wheeler: "Matter tells space-time how to curve, and curved space tells matter how to move."

Some comments: one can say that the basic idea is to change the geometry so that will incorporate gravitation. More precisely, to abandon "flat" 3 dimensional Euclidean space for 4-dimensional "curved" space.

This has the following consequence: in classical physics, the nature of gravitation had been a problem since Newton's time. In the words of A. Koyré [9] (page 16) "...in spite of the rational plausibility and mathematical simplicity of the Newtonian law...there was in it something that baffled the mind. Bodies attract each other, act upon each other (or at least behave as they did). But how do they manage to perform this action, to overcome the chasm of the void that so radically separates them and isolates them from each other? We must confess that nobody, not even Newton, could (or can) explain, or understand this *how*. Newton himself, as we well know, never admitted gravitation as a "physical" force."

These problems were swept by the incorporation of gravitation into the new geometry: there is no mysterious "gravitational" force acting on a particle: the particle just follows a geodesic! It is fair to say, nevertheless, that the extent to which Einstein's theory has solved this problem has been debated up to now.

The first two assumptions can be formulated in a straightforward way. The mathematical formulation of the third, i.e.,the field equations, is the real subject matter of this paper. A lot of work has to be done to carry out the program of the introduction. We do this for the vacuum only.

We reformulate Einstein's physical postulates as follows:

Postulate 1 [**Structure of space-time**]: Space-time is a D_∞-manifold M of dimension n with a symmetric connection ∇.

We leave the dimension of the space-time M indeterminate (rather than $n = 4$), since our derivation does not depend on the value of n.

Postulate 2 [**Law of motion**]: Freely falling particles near an attracting body describe geodesics. In other words, trajectories of particles under inertia and gravitation are solutions of the second order differential equations defining geodesics:

$$\gamma^{\bullet\bullet} = \nabla(\gamma^{\bullet}, \gamma^{\bullet})$$

or equivalently

$$\nabla_{\gamma^{\bullet}}\gamma^{\bullet} = 0$$

(cf. "affineconnections.pdf" page 12 in [12])

From here one can derive the second order differential equation for geodesics in terms of coordinates that can be found in texts of Differential Geometry (e.g. [14])

$$d^2x^k/dt^2 + \sum_{ij} \Gamma^k_{ij}(dx^i/dt)(dx^j/dt) = 0$$

(cf. "affineconnections.pdf" page 14 in [12].)

Imitating what we did for the Newtonian gravitational theory, we assume that space-time is a D_∞-manifold, that vector fields are maps $Q : U \longrightarrow U^D$ with U an infinitesimal neighborhood of M and trajectories, in particular geodesics, curves with domain D_∞.

Since a D_∞-manifold M is microlinear, each tangent space $\pi_M^{-1}(m)$ is a R-modules ($m \in M$). (Here $\pi_M : M^D \longrightarrow M$ is the evaluation at 0). A microlinear space *has dimension* n if all the tangent spaces are free R-modules of dimension n.

In the first section, Laplace field formulation of Newton's law of gravitation was stated in terms of the linear transformation $\Psi : E_x \longrightarrow E_x$ as $trace(\Psi) = 0$.

Trying to reproduce the computations of section 1 in our relativistic context, we take for G the second order DE of geodesics. Furthermore, we keep the same notion of neighbor. For the record:

A *neighbor of an integral curve* γ *of a vector field* X is a vector field W along γ such that the trajectories of W, namely the γ_h's defined by $\gamma_h(t) = W(t)(h)$ are integral curves of X.

The existence of neighbors need not be postulated, but can be proved

Proposition 4.1 (There are enough neighbors) *Let X be a vector field, γ an integral curve of X and $v \in M_x$. Then there is a unique neighbor W of γ in X with $W(0) = v$.*

Proof: cf. Section 5 (Proposition 5.1)

A G-field becomes a vector field whose integral curves are solutions of G, i.e., geodesics. We call such fields *geodesic fields*. Thus, a *geodesic field* is a vector field whose integral curves are trajectories of free falling particles, i.e., geodesics according to Postulate 2. (Recall that in the Newtonian framework, the integral curves of the gravitational field G are the trajectories of free falling particles).

As before, we do not postulate the existence of geodesic fields either, but prove their existence:

Proposition 4.2 (There are enough geodesic fields) *For every $u \in M_x$ not zero, there is a geodesic field G^u such that $G^u(x) = u$.*

Proof: In fact, G is "reducible" to a family of vector fields G^u indexed by non-zero vectors $u \in M_x$. Then G^u is the sought geodesic field. Cf. section 5 (Proposition 5.2)

Following the analogy with Newtonian gravitation we define a map

$$\Psi^u : M_x \longrightarrow M_x$$

for each non zero vector $u \in M_x$ as follows: let $\gamma : D_\infty \longrightarrow M$ be the integral curve of G^u with initial condition $\gamma(0) = x$.

Notice that $\gamma^\bullet(0) = G^u(\gamma(0)) = G^u(x) = u$.

Let now $v \in M_x$ be arbitrary. Then there is a unique neighbor W of γ in G^u with $W(0) = v$. Define

$$\Psi^u(v) = \nabla^2_{\gamma^\bullet} W(0)$$

We would like to formulate Einstein's vacuum field equation equation as

$$trace(\Psi^u) = 0$$

However, there are several problems with this formulation: first we have to prove that Ψ^u is R-linear. Furthermore,Ψ^u seems to depend not only on u

but on G^u and, worst of all, it is not informative, in fact, we would like to tie Ψ^u to the curvature of the space.

The following takes care of all of these problems

Theorem 4.3 (Main Theorem, cf. [13])

$$(\nabla^2_{\gamma^\bullet} W)(t) = R_{\gamma^\bullet(t)W(t)}\gamma^\bullet(t)$$

where R is the Riemann-Christoffel tensor.

NB The definition of this tensor, as well as all the properties that we require in our proofs can be found in "Riemann-Christoffel.pdf" in [12].

Corollary 4.4

$$\Psi(v) = R_{uv}u$$

To prove the theorem we need the following

Lemma 4.5 *5.4 Let $\gamma : D_\infty \longrightarrow M$ be an infinitesimal curve in M, $p = \gamma(0)$ and Q a vector field defined on an infinitesimal neighborhood of p with $Q(p) \neq 0$. Then for every neighbor W of γ in Q there is a vector field $\tilde{W}$ of M such that $[Q, \tilde{W}] = 0$ and $\tilde{W} \circ \gamma = W$.*

NB Definitions and properties of Lie Derivatives $L_Q W$ and Lie Brackets $[X, Y]$ can be found in "Liederivatives.pdf" in [12].

Proof: cf. Section 5 (Lemma 5.4)

Proof (of the main theorem): We recall the following results (cf. [10])

1. $\nabla_X Y = \nabla_Y X + [X, Y]$, provided that ∇ is symmetric
2. $R_{XY} Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}(Z)$

The rest of the proof is now a simple computation:

$$\begin{aligned}
\nabla^2_Q \tilde{W} &= \nabla_Q \nabla_Q \tilde{W} \\
&= \nabla_Q(\nabla_{\tilde{W}} Q + [Q, \tilde{W}]) \\
&= \nabla_Q \nabla_{\tilde{W}} Q \\
&= R_{Q,\tilde{W}} Q + \nabla_{\tilde{W}} \nabla_Q Q + \nabla_{[Q,\tilde{W}]} Q \\
&= R_{Q\tilde{W}} Q
\end{aligned}$$

Post-composing with γ, we obtain the desired result. In more details:

Claim 1: $\tilde{Q}_{\gamma(t)}(\delta) = \gamma(t+\delta)$ for every $\delta \in D_2$. Since M is microlinear, it is enough to show that $(Q_{d_1} \circ Q_{d_2})(\gamma(t)) = \gamma(t + d_1 + d_2)$. This is clear. In fact, since γ is an integral curve of Q,

$$Q_{d_1} \circ Q_{d_2}(\gamma(t)) = Q_{d_1}(Q_{d_2}(\gamma(t))) = Q_{d_1}(\gamma(t+d_2)) = \gamma(t+d_2+d_1)$$

Claim 2: $(\nabla_Q \tilde{W})(\gamma(t)) = (\nabla_{\gamma^\bullet} W)(t)$. This is proved by starting from claim 1 and the definition of $\nabla_X Y$ (cf. [11] page 200 and [12]).

$$\begin{aligned} h(\nabla_Q \tilde{W})(\gamma(t)) &= \nabla([d \mapsto \tilde{Q}_{\gamma(t)}(d+h)], \tilde{W}_{Q_{\gamma(t)}(h)})(-h) - \tilde{W}_{\gamma(t)} \\ &= \nabla([d \mapsto \gamma(t+h+d)], \tilde{W}_{\gamma(t+h)})(-h) - W(t) \\ &= \nabla(\gamma^\bullet(t+h), W(t+h))(-h) - W(t) \\ &= h(\nabla_{\gamma^\bullet} W)(t) \end{aligned}$$

The claim results from canceling the universally quantified h.

Claim 3: $(R_{Q\tilde{W}}Q)_{\gamma(t)} = R_{\gamma^\bullet(t)W(t)}\gamma^\bullet(t)$. This follows from the definition of the Riemann-Christoffel tensor, by noticing that $Q(\gamma(t)) = \gamma^\bullet(t) \in M_{\gamma(t)}$ and $\tilde{W}(\gamma(t)) = W(t) \in M_{\gamma(t)}$.

By analogy with Newton's gravitational theory, we postulate

Einstein's vacuum field equation [First Formulation]

$$trace(\Psi^u) = 0 \text{ for all non zero } u \in M_x$$

This is not the usual formulation of this equation, but it is equivalent.

First, we define the *Ricci tensor* as the contraction of the Riemann-Christoffel curvature tensor, i.e., in classical notation $R_{ki} = \sum_l R^l_{kil}$. From a more conceptual or invariant point of view, recall that the Riemann-Christoffel curvature tensor is given by a map $R : M^D \times_M M^D \times_M M^D \longrightarrow M^D$.

The contraction is a new map $Ric : M^D \times_M M^D \longrightarrow M^D$ defined as follows: if $u, v \in M_x$, let $\Psi(u,v) : M_x \longrightarrow M_x$ be the R-linear map defined by $\Psi(u,v)(w) = R_{uw}v$. Then $Ric(u,v) = trace(\Psi(u,v))$.

To connect this invariant definition of Ric with the classical expression of the Ricci tensor we first take any basis of M_x to represent $\Psi(u,v)$ as a matrix. By definition, $Ric(u,v)$ is the trace of this matrix. Take, for instance the basis $\{\partial/\partial x^i|_x\}_i$. The matrix of $\Psi(u,v)$ relative to this basis is $(\omega^i(\Psi(u,v)(\partial/\partial x^j|_x))_{ij}$, where $\{\omega^i\}_i$ is the dual basis, and its trace is $Ric(u,v) = \sum_i \omega^i(R_{u(\partial/\partial x^i|_x)}v)$.

Next, we start with the following notations and identifications

$$\begin{cases} R_{ij}^k = R_{\partial_i\partial_j}\partial_k \text{ (where } \partial_i = \partial/\partial x_i) \\ (R_{ij}^k)_l \text{ in our notation} = R_{kij}^l \text{ in Einstein's notation} \end{cases}$$

Thus $R_{ki} = \sum_l R_{kil}^l = \sum_l (R_{il}^k)_l = \sum_l \omega^l(R_{\partial_i\partial_l}\partial_k) = trace(\Psi(\partial_i,\partial_k)) = Ric(\partial_i,\partial_k)$

Proposition 4.6 *If ∇ is symmetric, then*

$$Ric(u,v) = Ric(v,u)$$

Proof: cf. "Metrics.pdf" page 8 in [12].

This allows us to re-write the first formulation of Einstein vacuum field equation as $Ric(u,u) = 0$ for every non zero $u \in M_x$, since $\Psi^u(v) = R_{uv}$ and hence $trace(\Psi^u) = Ric(u,u)$. But this implies that $Ric = 0$.

Indeed, let $v, w \in M_x$ be *arbitrary* vectors. Write $v = \sum v_i e_i$ and $w = \sum w_i e_i$. Then $Ric(u,v) = \sum_{ij} v_i w_j Ric(e_i,e_j)$. But $Ric(e_i+e_j, e_i+e_j) = Ric(e_i,e_i) + 2Ric(e_i,e_j) + Ric(e_j,e_j)$. By hypothesis, the LHS and the extreme terms of the RHS are 0, so the middle term must be zero, i.e., $Ric(e_i,e_j) = 0$. Thus, $Ric = 0$. In other words, we may reformulate the above equation as

Einstein's vacuum field equation [Final Formulation]

$$Ric = 0$$

which is the usual form of this equation.

5 Missing Proofs

In this final section we provide proofs for the lemma and propositions required to prove the main theorem

Proposition 5.1 (There are enough neighbors) *Let X be a vector field, γ an integral curve of X and $v \in M_x$ (with $\gamma(0) = x$). Then there is a unique neighbor V of γ in X with $V(0) = v$.*

Proof: Let $\gamma : D_\infty \longrightarrow M$ an integral curve of X and $u \in M_x$. Define $V : D_\infty \longrightarrow M^D$ by $V(t)(h) = \gamma_h(t)$, where γ_h is the unique integral curve of X with initial condition $\gamma_h(0) = v(h)$.

Thus, $V(t)(0) = \gamma_0(t) = \gamma(t)$, since both γ and γ_0 are integral curves of Q with the same initial condition. In other words, V is a vector field along γ. Furthermore, $L_X V = 0$, by construction. Finally, $V(0)(h) = \gamma_h(0) = u(h)$, and this shows that $V(0) = u$.

To show uniqueness, assume that V^1 and V^2 are two such. Then V_h^1 and V_h^2 satisfy the same (first-order) differential equation, namely X, with the same initial conditions since $V_h^1(0) = V^1(0)(h) = v(h) = V^2(0)(h) = V_h^2(0)$. Thus, they coincide.

The following generalizes a result of [13]:

Proposition 5.2 (Reduction theorem) *Let Y be an autonomous second order equation, i.e., a symmetric vector field $Y : M^D \longrightarrow (M^D)^D$. Then there is a family Y^u of vector fields indexed by non zero vectors $u \in M_x$ with the following property: if $\gamma : D_\infty \longrightarrow R$ is a curve with $\gamma(0) = x$ and $\gamma^\bullet(0) = u$,*

$$\gamma \textit{ is a solution of } Y \textit{ iff } \gamma \textit{ is an integral curve of } Y^u$$

We say that Y is *reducible* to the family $(Y^u)_u$.

Call a vector field Q an $Y - field$ iff all integral curves of Q are solutions of Y.

Corollary 5.3 *All the Y^u are Y-fields.*

Proof of the Proposition: By choosing appropriate coordinates we may assume that $x = (0, \ldots, 0)$ and $u = (0, \ldots, 0, (0, \ldots 0, u_0))$ with u_0 invertible.

Define a map $\phi : D_\infty^n \longrightarrow D_\infty^n$ by the formula

$$\phi(\delta^1, \ldots, \delta^{n-1}, t) = \gamma_{(\delta^1, \ldots, \delta^{n-1}, 0)}(t)$$

where $\gamma_{(\delta^1, \ldots, \delta^{n-1}, 0)}$ is the solution of the equation Y starting at $(\delta^1, \ldots, \delta^{n-1}, 0)$ with initial "velocity" u, i.e.,

$$\begin{cases} \gamma_{(\delta^1, \ldots, \delta^{n-1}, 0)}(0) & = & (\delta^1, \ldots, \delta^{n-1}, 0) \\ \gamma^\bullet_{(\delta^1, \ldots, \delta^{n-1}, 0)}(0) & = & (\delta^1, \ldots, \delta^{n-1}, 0, (0, \ldots, 0, u_0)). \end{cases}$$

(For the existence and uniqueness of these solutions, see "Second order DE and geodesics.pdf" in [12] and [8]).

We claim that ϕ is invertible:

Write $\phi = (\phi^1, \ldots, \phi^n)$. Spelling the initial conditions of $\gamma_{(\delta^1,\ldots,\delta^{n-1},t)}$ in terms of the components of ϕ we get

$$\begin{cases} \phi^i(\delta^1, \ldots, \delta^{n-1}, 0) = \delta^i \quad (i < n) \\ \phi^n(\delta^1, \ldots, \delta^{n-1}, 0) = 0 \end{cases}$$

for the first and

$$\begin{cases} \partial\phi^i/\partial t(\delta^1, \ldots, \delta^{n-1}, 0) = 0 \quad (i < n) \\ \partial\phi^n/\partial t(\delta^1, \ldots, \delta^{n-1}, 0) = u_0 \end{cases}$$

for the second.

By developing in Taylor series,

$$\begin{aligned} \phi^i(\delta^1, \ldots, \delta^j + d, \ldots, \delta^{n-1}, 0) &= \phi^i(\delta^1, \ldots, \delta^j, \ldots, \delta^{n-1}, 0) \\ &+ d(\partial\phi^i/\partial y^j)(\delta^1, \ldots, \delta^j, \ldots, \delta^{n-1}, 0) \end{aligned}$$

By identifying corresponding terms in this equation,

$$\begin{cases} (\partial\phi^i/\partial y^j)(\delta^1, \ldots, \delta^j, \ldots, \delta^{n-1}, 0) = 0 \quad (i \neq j) \\ (\partial\phi^i/\partial y^i)(\delta^1, \ldots, \delta^j, \ldots, \delta^{n-1}, 0) = 1 \end{cases}$$

Therefore $\phi(0, \ldots, 0, 0) = (0, \ldots, 0, 0)$ and the Jacobian of ϕ at $(0, \ldots, 0, 0)$ is $u_0 \neq 0$. The claim is then a consequence of the "Infinitesimal Inverse function theorem.pdf" in [12].

We let ψ be the inverse of ϕ.

Define the vector field Y^u or simply Q by the formula

$$Q(x^1, \ldots, x^n) = \gamma^\bullet_{(\delta^1,\ldots,\delta^{n-1},0)}(t)$$

with $\phi(\delta^1, \ldots, \delta^{n-1}, t) = (x^1, \ldots, x^n)$.

Claim 1: Q is a vector field and $Q(x) = u$.

The fact that Q is a vector field is obvious:

$$\begin{array}{rcl} Q(x^1,\ldots,x^n)(0) & = & \gamma^{\bullet}_{(\delta^1,\ldots,\delta^{n-1},0)}(t)(0) \\ & = & \gamma_{(\delta^1,\ldots,\delta^{n-1},0)}(t) \\ & = & \phi(\delta^1,\ldots,\delta^{n-1},t) \\ & = & (x^1,\ldots,x^n) \end{array}$$

On the other hand,

$$Q(x) = Q(0,\ldots,0) = \gamma^{\bullet}_{(0,\ldots,0,0)}(0) = (0,\ldots,0,(0,\ldots,0,u_0)) = u$$

by the second initial condition that γ satisfies.

Claim 2: The $\gamma_{(\delta^1,\ldots,\delta^{n-1},0)}(-)$'s are integral curves of Q, i.e.,

$$\gamma^{\bullet}_{(\delta^1,\ldots,\delta^{n-1},0)}(t) = Q(\gamma_{(\delta^1,\ldots,\delta^{n-1},0)}(t)$$

In fact, let $\gamma_{(\delta^1,\ldots,\delta^{n-1},0)}(t) = (x^1,\ldots,x^n)$. We saw that, $\gamma_{(\delta^1,\ldots,\delta^{n-1},0)}(t) = \phi(\delta^1,\ldots,\delta^{n-1},t)$. On the other and, by definition of Q,

$$Q(\gamma_{(\delta^1,\ldots,\delta^{n-1},0)}(t)) = Q(x^1,\ldots,x^n) = \gamma^{\bullet}_{(\delta^{1\prime},\ldots,\delta^{n-1\prime},0)}(t')$$

where $\phi(\delta^{1\prime},\ldots,\delta^{n-1\prime},t') = (x^1,\ldots,x^n)$.

Since $\phi(\delta^1,\ldots,\delta^{n-1},t) = \gamma_{(\delta^1,\ldots,\delta^{n-1},0)}(t) = (x^1,\ldots,x^n)$ and ϕ is a bijection, $\delta^i = \delta^{i\prime}$ and $t = t'$, completing the proof.

Claim 3: The integral curves of Q are precisely those of the form $\delta(t) = \gamma_{(x^1,\ldots,x^{n-1},0)}(x^n+t)$, for $(x^1,\ldots,x^n) \in D^n_\infty$.

In fact, all curves of this form are integral curves of Q : let $u(t) = x^n + t$ and $\gamma(t) = \gamma_{(x^1,\ldots,x^{n-1},0)}(t)$. Then $\delta = \gamma \circ u$.

We claim that

$$\begin{cases} \gamma^{\bullet} \circ u = (\gamma \circ u)^{\bullet} = \delta^{\bullet} \\ \gamma^{\bullet\bullet} \circ u = (\gamma \circ u)^{\bullet\bullet} = \delta^{\bullet\bullet} \end{cases}$$

Indeed, computing the first term of the first equation at (t,d) with $d \in D$, we have

$$\begin{array}{rcl} \gamma^{\bullet}(u(t))(d) & = & \gamma(u(t)+d) \\ & = & \gamma((x^n+t)+d) \\ & = & \gamma(x^n+(t+d)) \\ & = & \gamma(u(t+d)) \\ & = & (\gamma \circ u)(t+d) \\ & = & (\gamma \circ u)^{\bullet}(t)(d) \end{array}$$

The second equation is proved by iteration.

As a consequence, we have the two equations

$$\begin{cases} \delta^\bullet = (\gamma \circ u)^\bullet = \gamma^\bullet \circ u = (Q \circ \gamma) \circ u = Q \circ (\gamma \circ u) = Q \circ \delta \\ \delta^{\bullet\bullet} = \gamma^{\bullet\bullet} \circ u = (Y \circ \gamma^\bullet) \circ u = Y \circ (\gamma^\bullet \circ u) = Y \circ \delta^\bullet \end{cases}$$

The first shows that curves of the stated form are integral curves of Q.

To prove the converse, let γ be an integral curve of Q with initial condition $\gamma(0) = (x_0^1, \ldots, x_0^n)$. Since ϕ is bijective, there is a unique n-tuple $(\delta^1, \ldots, \delta^n)$ such that $\phi(\delta^1, \ldots, \delta^n) = (x_0^1, \ldots, x_0^n)$. Define $\delta(t) = \gamma_{(\delta^1, \ldots, \delta^{n-1}, 0)}(\delta^n + t)$

Then, $\delta = \gamma$ since both are integral curves of Q with the same initial condition: $\delta(0) = \gamma_{(\delta^1, \ldots, \delta^{n-1}, 0)}(\delta^n) = \phi(\delta^1, \ldots, \delta^n) = (x_0^1, \ldots, x_0^n) = \gamma(0)$.

Proof of the Corollary: We have to show that if γ is a solution of Y, and $x^n \in D_\infty$, so is δ, where $\delta(t) = \gamma(x^n + t)$. This is a consequence of the second equation.

Lemma 5.4 *Let $\gamma : D_\infty \longrightarrow M$ be an infinitesimal curve in M, $p = \gamma(0)$ and Q a vector field defined on an infinitesimal neighborhood of p with $Q(p) \neq 0$. Then for every neighbor W of γ in Q there is a vector field $\tilde{W}$ of M such that $[Q, \tilde{W}] = 0$ and $\tilde{W} \circ \gamma = W$.*

NB Definitions and properties of Lie Derivatives ($L_Q W$) and Lie Brackets ($[Q, \tilde{W}]$) used in this proof can be found in “Liederivatives.pdf” in [12].

Proof: Since $Q(p) \neq 0$, by a change of coordinates we may assume that, locally around p, $Q = \partial/\partial x^1$ (see “Straightening out Theorem.pdf” in [12]). Write $W(t) = \sum_i w^i(t) \partial/\partial x^i|_{\gamma(t)}$. Then

$$0 = L_Q W(t) = \sum_i w^i(t) L_Q(\partial/\partial x^i|_{\gamma(t)}) + \sum_i (w^i)'(t) \partial/\partial x^i|_{\gamma(t)}.$$

Now, $L_Q((\partial/\partial x^i) \circ \gamma) = L_{\partial/\partial x^1}((\partial/\partial x^i) \circ \gamma) = [\partial/\partial x^1, \partial/\partial x^i] \circ \gamma$. But $[\partial/\partial x^i, \partial/\partial x^j] = 0$. This follows from the definition of Lie bracket

$$[\partial/\partial x^i, \partial/\partial x^j]_x(d_1 d_2) = (\partial/\partial x^j)_{-d_2} \circ (\partial/\partial x^i)_{-d_1} \circ (\partial/\partial x^j)_{d_2} \circ (\partial/\partial x^i)_{d_1}(x)$$

and the definition of the vector field $(\partial/\partial x^i)$, namely $(\partial/\partial x^i)_d(x) = x + de_i$.

Since $\{\partial/\partial x^i|_{\gamma(t)}\}_i$ is a basis, $(w^i)'(t) = 0$, i.e. $w^i(t) = a^i$ is a constant. Define

$$\tilde{W}_x = \sum_i a^i \partial/\partial x^i|_x.$$

This vector field has the required properties. Indeed, obviously $\tilde{W} \circ \gamma = W$. The second property is a consequence of bi-linearity of $[-,-]$

$$[Q, \tilde{W}] = [\partial/\partial x^1, \tilde{W}] = \sum_i a_i [\partial/\partial x^1, \partial/\partial x^i] = 0.$$

6 Appendix

6.1 A reformulation of Laplace equation

From what we did in the first section, we have an intuitive reformulation of Laplace equation due to Sachs/Wu that we now present. This follows from the following

Proposition 6.1 *Let $\Phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be any linear transformation. Then*

$$(1/4\pi) \int_{u \in S^2} \Phi(u) \cdot u \sigma = (1/3) Trace(\Phi)$$

where σ is the volume form of S^2.

Proof: Compute the integral using spherical coordinates:

$$\begin{cases} x_1 = cos\ \phi_1 \\ x_2 = sin\ \phi_1\ cos\ \phi_2 \\ x_3 = sin\ \phi_1\ sin\ \phi_2 \end{cases}$$

$(0 \leq \phi_1 \leq \pi,\ 0 \leq \phi_2 < 2\pi)$

In these coordinates, the volume form of S^2, namely $\sigma = u_1 du_2 \wedge du_3 + u_2 du_3 \wedge du_1 + u_3 du_1 \wedge du_2$ (see e.g. [1] page 96) becomes $\sigma = sin\ \phi_1 d\phi_1\ d\phi_2$

Let $(a_{ij})_{ij}$ be a matrix representation of Φ. Then $\Phi(u) \cdot u = \sum_{i,j} a_{ij} u_i u_j$ and a simple computation shows that

$$\int_{u \in S^2} u_i u_j \sigma = \begin{cases} 0 & \text{if } i \neq j \\ 4\pi/3 & \text{otherwise} \end{cases}$$

Thus, $\int_{u \in S^2} \Phi(u) \cdot u\sigma = (4\pi/3) \sum_i a_{ii} = (4\pi/3)\ Trace(\Phi)$ $\square$

In particular, this proposition applies to the linear transformation Φ given by the matrix $(\partial^2\phi/\partial x^\alpha x^\beta)_{\alpha\beta}$. In this case, there is a simple intuitive interpretation of this equation given in [13]: "Assume that you are in the middle of a cabin freely falling near the earth surface with lots of apples around you. Look at those apples situated at the same distance (which we assume small) from you and measure their acceleration towards (or away from) you. Their average is 0."

In our context, we may formulate this intuitive interpretation as a precise mathematical statement by defining "small" as being of square 0, i.e., an element of D.

Recall from the section on Newton's theory of gravitation that the difference between the acceleration of the apple whose trajectory is γ_h and yours (trajectory γ) at time 0 is $h\Psi(u)$.

We define the average or mean acceleration of all the apples at distance $h \in D$ from 0 to be

$$(1/4\pi) \int_{u \in S^2} h\Psi(u) \cdot u\sigma$$

where $\sigma = x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge x_3 + x_3 dx_1 \wedge dx_2$ is the volume form of S^2. Notice that since $\int_{S^2} \sigma = 4\pi$, the term "mean" is justified.

Applying the previous proposition to Ψ we have the following

Corollary 6.2 *The mean relative acceleration of all particles situated at distance h from 0 is*

$$(1/4\pi) \int_{u \in S^2} h\Psi(u) \cdot u\sigma = (h/3)\ trace(\Psi)$$

where $\sigma = x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge x_3 + x_3 dx_1 \wedge dx_2$ is the volume form of S^2.

The Sachs and Wu interpretation is the statement that the integral in the corollary is 0 for every $h \in D$. But this is equivalent to the statement that $\nabla^2\phi = 0$, since $\Psi = -\Phi$ and $trace(\Phi) = \nabla^2\phi$.

6.2 The first two postulates of Einstein

The aim of this section is to show how to derive postulates 1 and 2 from other principles discussed in the literature.

Following Einstein, space-time is assumed to be a 4-dimensional semi-Riemannian manifold M with metric g. Recall that a *semi-Riemannian manifold* is a manifold with a metric g having the property that around each point p there are local coordinates ξ with

$$(g_\xi)_{\alpha\beta}(p) = \eta_{\alpha\beta}$$

where the η's are the entries of the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In particular, $(\Gamma_\xi)^\gamma_{\alpha\beta}(p) = 0$ (see "metrics.pdf" corollary 0.7 page 6 in [12]).

Ad Postulate 1: From g we can manufacture a symmetric connexion ∇. (Details in "metrics.pdf" in [12].)

The unique symmetric connection compatible with a given metric is called, improperly according to Spivak ([14] II page 256), the *Levi-Civita connection* for the metric.

Ad Postulate 2: This axiom is introduced as such in General Relativity under the name "Equation of motion" and it is deduced/motivated by the so-called *equivalence principle* (see Einstein [2] and Weinberg [15]).

I will not go into the questions of validity of this principle (see V.Fock [6]), or the shortcomings of Weinberg's "proof", but point out that Postulate 2 can be deduced from a very weak "point-wise" form of the equivalence principle that seems un-objectionable and may be formulated as follows:

Let $\gamma : D_\infty \longrightarrow M = R^n$ be the trajectory of a free falling particle and p a point of γ. Whenever ξ are local geodesic coordinates around p (i.e., with $\Gamma^\delta_{\alpha\beta}(p) = 0$), the trajectory satisfies

$$(d^2\xi^k/dt^2)(p) = 0$$

at the point p.

From this principle we deduce that γ is indeed a geodesic with equation

$$d^2x^k/dt^2 + \sum_{\alpha,\beta} \Gamma^k_{\alpha\beta}(dx^\alpha/dt)(dx^\beta/dt) = 0$$

An indication of the proof: since ξ^k is a function of the x^α, a simple application of the chain rule gives (at p)

$$0 = \sum_\alpha (\partial\xi^k/\partial x^\alpha)(d^2x^\alpha/dt^2) + \sum_{\alpha,\beta}(\partial^2\xi^k/\partial x^\alpha\partial x^\beta)(dx^\alpha/dt)(dx^\beta/dt)$$

By multiplying this equation by $\partial x^\alpha/\partial\xi^k$ and adding with respect to α, we obtain

$$d^2x^k/dt^2 + \sum_{\alpha\beta}(\Gamma^k_{\alpha\beta})_{x\xi}(dx^\alpha/dt)(dx^\beta/dt) = 0$$

since ξ and x are inverse of each other.

By applying the transformation rule for Γ at point p, $(\Gamma^k_{\alpha\beta})_{x\xi} = (\Gamma^k_{\alpha\beta})_x$. Since p was arbitrary, this concludes the proof.

NB Recall that going from coordinates x to x', Γ is transformed as follows

$$\begin{array}{rcl} (\Gamma^\gamma_{\alpha\beta})_{x'} & = & \sum_{ijk}(\Gamma^k_{ij})_x(\partial x^i/\partial x'^\alpha)(\partial x^j/\partial x'^\beta)(\partial x'^\gamma/\partial x^k) \\ & + & \sum_i(\partial^2 x^i/\partial x'^\alpha\partial x'^\beta)(\partial x'^\gamma/\partial x^i) \end{array}$$

(cf “Affineconnections.pdf” in [12]).

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