Infinitesimal version of the Inverse Function Theorem

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The aim of this pamphlet is to prove the following

Theorem 0.1 *Let* $f : D^n_\infty \longrightarrow D^n_\infty$ *be a function such that* $f(0) = 0$ *and* $f^\bullet(0)$ *is invertible. Then* f *is an isomorphism.*

Proof: We do first the case $n = 3$: Write $f = (f^1, f^2, f^3)$ and develop the f^i's in Taylor series:

$$(*) \quad \begin{cases} f^1 = a^1_{(1,0,0)}x_1 + a^1_{(0,1,0)}x_2 + a^1_{(0,0,1)}x_3 + a^1_{(2,0,0)}x_1^2 + a^1_{(1,1,0)}x_1x_2 + \dots \\ f^2 = a^2_{(1,0,0)}x_1 + a^2_{(0,1,0)}x_2 + a^2_{(0,0,1)}x_3 + a^2_{(2,0,0)}x_1^2 + a^2_{(1,1,0)}x_1x_2 + \dots \\ f^3 = a^3_{(1,0,0)}x_1 + a^3_{(0,1,0)}x_2 + a^3_{(0,0,1)}x_3 + a^3_{(2,0,0)}x_1^2 + a^3_{(1,1,0)}x_1x_2 + \dots \end{cases}$$

We want to find $g = (g_1, g_2, g_3)$ such that $f \circ g = id$. By developing in Taylor series

$$(**) \quad \begin{cases} g_1 = b^1_{(1,0,0)}x_1 + b^1_{(0,1,0)}x_2 + b^1_{(0,0,1)}x_3 + b^1_{(1,1,0)}x_1x_2 + \dots \\ g_2 = b^2_{(1,0,0)}x_1 + b^2_{(0,1,0)}x_2 + b^2_{(0,0,1)}x_3 + b^2_{(1,1,0)}x_1x_2 + \dots \\ g_3 = b^3_{(1,0,0)}x_1 + b^3_{(0,1,0)}x_2 + b^3_{(0,0,1)}x_3 + b^3_{(1,1,0)}x_1x_2 + \dots \end{cases}$$

Writing f and g in terms of its components, the composite $f \circ g$ may be written as

$$(***) \quad \begin{cases} f^1(g_1, g_2, g_3) = a^1_{(1,0,0)}g_1 + a^1_{(0,1,0)}g_2 + a^1_{(0,0,1)}g_3 + \\ a^1_{(2,0,0)}(g_1)^2 + a^1_{(1,1,0)}g_1g_2 + \dots \\ f^2(g_1, g_2, g_3) = a^2_{(1,0,0)}g_1 + a^2_{(0,1,0)}g_2 + a^2_{(0,0,1)}g_3 + \\ a^2_{(2,0,0)}(g_1)^2 + a^2_{(1,1,0)}g_1g_2 + \dots \\ f^3(g_1, g_2, g_3) = a^3_{(1,0,0)}g_1 + a^3_{(0,1,0)}g_2 + a^3_{(0,0,1)}g_3 + \\ a^3_{(2,0,0)}(g_1)^2 + a^3_{(1,1,0)}g_1g_2 + \dots \end{cases}$$

The equation $f \circ g = id$ may be re-written in terms of components as

$$f^i(g_1(x_1, x_2, x_3), g_2(x_1, x_2, x_3), g_3(x_1, x_2, x_3)) = x_i$$

The coefficient of $x_1 = x_1^{(1,0,0)}$ in the LHS is

$$a^1_{(1,0,0)} b^1_{(1,0,0)} + a^1_{(0,1,0)} b^2_{(1,0,0)} + a^1_{(0,0,1)} b^3_{(1,0,0)}$$

The coefficient of $x_1^2 = x_1^{(2,0,0)}$ in the LHS is

$$\begin{array}{l} a^1_{(1,0,0)} b^1_{(2,0,0)} + a^1_{(0,1,0)} b^2_{(2,0,0)} + a^1_{(0,0,1)} b^3_{(2,0,0)} + \\ a^1_{(2,0,0)} (b^1_{(1,0,0)})^2 + a^1_{(1,1,0)} b^1_{(1,0,0)} b^2_{(1,0,0)} + a^1_{(0,1,1)} b^2_{(1,0,0)} b^3_{(1,0,0)} + a^1_{(1,0,1)} b^3_{(1,0,0)} b^1_{(1,0,0)} \end{array}$$

The coefficient of $x_1 x_2 = x^{(1,0,0)} x^{(0,1,0)} = x^{(2,0,0)}$ in the LHS is

$$\left\{ \begin{array}{r} a^1_{(1,0,0)} b^1_{(1,1,0)} + a^1_{(0,1,0)} b^2_{(1,1,0)} + a^1_{(0,0,1)} b^3_{(1,1,0)} + \\ a^1_{(1,0,0)} [b^1_{(1,0,0)} b^2_{(0,1,0)} + b^1_{(0,1,0)} b^2_{(1,0,0)}] + \\ a^1_{(1,0,1)} [b^1_{(1,0,0)} b^3_{(0,1,0)} + b^1_{(0,1,0)} b^3_{(1,0,0)}] + \\ a^1_{(0,1,1)} [b^2_{(1,0,0)} b^3_{(0,1,0)} + b^2_{(0,1,0)} b^3_{(1,0,0)}] + \\ 2[a^1_{(2,0,0)} b^1_{(1,0,0)} b^1_{0,1,0)} + a^1_{(,2,0)} b^2_{(1,0,0)} b^2_{0,1,0)} + a^1_{(0,0,2)} b^3_{(1,0,0)} b^3_{0,1,0)}] \end{array} \right.$$

For the general case, the crucial observation is the following rather obvious

Lemma 0.2 *The coefficient of x^α in $f^j(g_1(x), \ldots, g_n(x))$ is an expression of the form*

$$\sum_{i=1}^{n} a^j_{e_i} b^i_\alpha + \text{ terms in } b^i_\beta$$

with $|\beta| < |\alpha|$. (Here, $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 in the i^{th} place.)

We now compare coefficients of $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ in both sides, with α the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. For instance, for $x^{(1,0,0)} = x_1^1 x_2^0 x_3^0 = x^1$,

$$\left\{ \begin{array}{l} 1 = a^1_{(1,0,0)} b^1_{(1,0,0)} + a^1_{(0,1,0)} b^2_{(1,0,0)} + a^1_{(0,0,1)} b^3_{(1,0,0)} \\ 0 = a^2_{(1,0,0)} b^1_{(1,0,0)} + a^2_{(0,1,0)} b^2_{(1,0,0)} + a^2_{(0,0,1)} b^3_{(1,0,0)} \\ 0 = a^3_{(1,0,0)} b^1_{(1,0,0)} + a^3_{(0,1,0)} b^2_{(1,0,0)} + a^3_{(0,0,1)} b^3_{(1,0,0)} \end{array} \right.$$

Since the Jacobian is invertible, $b^1_{(1,0,0)}, b^2_{(1,0,0)}, b^3_{(1,0,0)}$ are uniquely determined by these equations. Proceeding as before with the coefficients for x_2 (respectively x_3) we show, similarly, that $b^1_{(0,1,0)}, b^2_{(0,1,0)}, b^3_{(0,1,0)}$ (respectively

$b^1_{(0,0,1)}, b^2_{(0,0,1)}, b^3_{(0,0,1)}$) are also uniquely determined. We keep on going by identifying further coefficients. We do this by course of value induction: assume that all b^i_β are determined for all β with $|\beta| < |\alpha|$. Then b^i_α is determined. As an example: by comparing the coefficients of x_1^2 in (***), we obtain

$$\begin{cases} a^1_{(1,0,0)} b^1_{(2,0,0)} + a^1_{(0,1,0)} b^2_{(2,0,0)} + a^1_{(0,0,1)} b^3_{(2,0,0)} = C_1 \\ a^2_{(1,0,0)} b^1_{(2,0,0)} + a^2_{(0,1,0)} b^2_{(2,0,0)} + a^2_{(0,0,1)} b^3_{(2,0,0)} = C_2 \\ a^3_{(1,0,0)} b^1_{(2,0,0)} + a^3_{(0,1,0)} b^2_{(2,0,0)} + a^3_{(0,0,1)} b^3_{(2,0,0)} = C_3 \end{cases}$$

with

$$-C_1 = a^1_{(2,0,0)}(b^1_{(1,0,0)})^2 + a^1_{(1,1,0)} b^1_{(1,0,0)} b^2_{(1,0,0)} + \\ a^1_{(0,1,1)} b^2_{(1,0,0)} b^3_{(1,0,0)} + a^1_{(1,0,1)} b^3_{(1,0,0)} b^1_{(1,0,0)}$$

already determined. Similarly for C_2 and C_3.

Since the Jacobian is invertible, $b^1_{(2,0,0)}, b^2_{(2,0,0)}, b^3_{(2,0,0)}$ are uniquely determined.

In a similar vein, by comparing coefficients of x_1x_2 we can deduce that $b^1_{(1,1,0)}, b^2_{(1,1,0)}, b^3_{(1,1,0)}$ are uniquely determined too, etc.

In the general case, the previous lemma implies the conclusion: in fact, assume by course of value induction that all b^i_β's are uniquely determined. By comparing coefficients in $f \circ g = id$, we have the system of equations to determine the coefficient of x_α, in $f^j(g_1, \dots g_n)$:

$$\delta^{e_j}_\alpha = \sum_{i=1}^n a^j_{e_i} b^i_\alpha + \text{ terms in } b^i_\beta$$

or, isolating the sum,

$$\sum_{i=1}^n a^j_{e_i} b^i_\alpha = C^j = \text{a constant}$$

since all the terms b^i_β's are uniquely determined. But the Jacobian is invertible and this implies that all the b^j_α (all j) are uniquely determined. COMPLETE AND CHECK!