Babbage functional equation

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The aim of this note is to find all continuous everywhere defined solutions of the Babbage functional equation

$$(^*) \quad f^n = id$$

where $n \geq 1$, $id(x) = x$ and f^n is defined recursively by

$$\begin{cases} f^1 = f \\ f^{n+1} = f \circ f^n \end{cases}$$

From now on, by "solution" of $(^*)$ we mean a real, continuous, everywhere defined function f such that $f^n = id$.

A key observation of Babbage was that if f is a solution of $(^*)$ and ϕ a (real) continuous bijection of the reals, the conjugate $F = \phi^{-1} \circ f \circ \phi$ is again a solution of $(^*)$. Apparently, he believed that the general solution was obtained as the conjugate of a particular solution, on the basis that the conjugate depended on one function.

The theorem below shows that, in some sense, he was right.

Theorem 0.1 *If n is odd, the only solution of Babbage equation $(^*)$ is the identity. If n is even, the only solutions of Babbage equation $(^*)$ are the identity and the conjugates of $-id$.*

To prove the theorem we need several lemmas and propositions:

Lemma 0.2 *If f is a solution of $(^*)$, then f is a bijection.*

Proof: If $n = 1$ there is nothing to prove. Let $n > 1$ and $g = f^{n-1}$. Then $f \circ g = g \circ f = id$. This obviously implies the conclusion. Take, for instance, the injectivity of f: if $f(a) = f(b)$, $a = g(f(a)) = g(f(b)) = b$.

Lemma 0.3 *If f is a solution of $f^n = id$ with $n \geq 1$, f is either monotonically increasing or monotonically decreasing.*

Proof: This is geometrically obvious, since an increasing function, say, can not start to decrease without taking a previous value, something that cannot happen if the function is injective. Analytically, this is a consequence of the intermediate value theorem.

The following simple observations will be very helpful:

Proposition 0.4 *(1) There are no monotonically decreasing solutions of the Babbage equation (*) if n is odd. (2) There are no solutions at all for the "dual Babbage functional equation" $f^m = -id$ if m is even.*

Proof: As for the first, assume that there is such an f and let $x > 0$. Then we have the chain of implications

$$\begin{array}{l} x > 0 \\ f(x) < f(0) \\ f^2(x) > f^2(0) \\ f^3(x) < f^3(0) \\ \ldots\ldots\ldots\ldots \\ f^n(x) < f^n(0) \end{array}$$

But this last inequality says that $x < 0$, a contradiction.

As for the second, the same proof works for the non existence of monotonically decreasing solutions of f^m for m even. On the other hand, if f is any monotonically increasing function, so is f^m, whereas $-id$ is monotonically decreasing. These functions cannot be equal.

Proposition 0.5 *Assume that f is a monotonically increasing solution of the Babbage equation $f^n = id$, with $n \geq 1$. Then $f = id$.*

Proof: Assume not. Then there is x_0 such that $f(x_0) \neq x_0$. Then either $x_0 < f(x_0)$ or $f(x_0) < x_0$. In the first case, we have the following chain of implications

$$\begin{array}{l} x_0 < f(x_0) \\ f(x_0) < f^2(x_0) \\ \dots\dots\dots\dots\dots \\ f^{n-1}(x_0) < f^n(x_0) = x_0 \end{array}$$

Thus, $x_0 < x_0$, a contradiction. The other case is similar, proving the proposition.

NB A constructive proof for $n = 2$, avoiding the argument by contradiction, has been given by A. Royer [3], completing an argument of Lévy-Leblond [1]. I give this proof, in my own version, in the Appendix.

From all of this, the first part of the theorem follows immediately:

Corollary 0.6 *Assume that n is odd. Then the only solution of $f^n = id$ is the identity function.*

We now prove the second part of the theorem, by first showing the particular case $n = 2$:

Lemma 0.7 *The only solutions of the Babbage equation $f^2 = id$ are the id and the conjugates of $-id$*

Proof: Let f be a solution. If f is monotonically increasing, then $f = id$ by proposition 0.5. Assume that f is monotonically decreasing. The proof proceeds in several steps:

(i) f has a unique fixed point: define

$$\begin{array}{l} U = \{x | x < f(x)\} \\ V = \{x | x > f(x)\} \end{array}$$

If there are no fixed points, then $U \cup V = R$. Since U and V are open and disjoint, $U = R$ or $V = R$.

Suppose that $U = R$. Assume $x \in R$. Then $x < f(x)$ and $f(x) < f(f(x))$. Therefore $x < x$, a contradiction. Similarly $V = R$ implies a contradiction. Therefore f has at least one fixed point x_1. (Notice that this is independent of the fact that f is monotonically decreasing). If f is monotonically decreasing, then x_1 is the only fixed point. In fact, let x_2 be another. We may assume that $x_1 < x_2$. Therefore $f(x_1) > f(x_2)$, i.e., $x_1 > x_2$, a contradiction. Thus $x_2 = x_1$.

The unique fixed point of f divides R into two intervals plus one point: the first $(-\infty, x_1)$, the second (x_1, ∞) and the point x_1.

Define $\phi(x_1) = 0$ and $\phi : (x_1, \infty) \longrightarrow R$ to be any monotonically increasing non-negative continuous that tends to 0 when x tends to x_1 from the right and to ∞ when x tends to ∞.

The question is to define $\phi : (-\infty, x_1) \longrightarrow R$.

We recall that we would like to have $f(x) = \phi^{-1}(-\phi(x))$ or, equivalently, $\phi(f(x)) = -\phi(x)$. Assume that $x < x_1$. Then $f(x) > f(x_1) = x_1$. Thus, $\phi(f(x))$ has already been defined and we can simply let

$$\phi(x) = -\phi(f(x))$$

We have to show several things:

(ii) ϕ is a continuous bijection.

The fact that is ϕ continuous for all $x \neq x_1$ is clear since both restrictions $\phi_{(x_1,\infty)}$ and $\phi_{(-\infty,x_1)}$ are continuous. Furthermore $\phi(x)$ tends to 0 whether we come from the right of x_1 (by definition of ϕ) or from the left, since in this case $\phi(x) = -\phi(f(x))$ tends to $-\phi(f(x_1)) = -\phi(x_1) = 0$. Thus, ϕ is also continuous at x_1.

(ii)a: ϕ is an injection. Assume that $\phi(a) = \phi(b)$. Then both a and b must be in the same interval (ϕ on one interval is non-negative and negative on the other. If both are in the right interval, then $a = b$ by definition of ϕ. Assume, then, that both are in the left and that $\phi(a) = \phi(b)$. Then $\phi(a) = -\phi(f(a)) = -\phi(f(b)) = \phi(b)$. Thus, $\phi(f(a)) = \phi(f(b))$ and hence $f(a) = f(b)$ (since both $f(a)$ and $f(b)$ are in the second interval). Since f is injective, $a = b$.

(ii)b: ϕ is surjective. This is obvious: it is enough to observe that if a sequence $\{x_n\}$ is in the second interval and tends to ∞, $\phi(x_n)$ tends to infinity and $-f(\phi(x_n))$ tends to $-\infty$.

Finally, we have to check that $\phi(x) = -\phi(f(x))$. If x is in the first interval, this is true by definition. Assume then that x is in the second interval, i.e., $x > x_1$. Then $f(x) < f(x_1) = x_1$ and $\phi(f(x)) = -\phi(f(f(x)) = -\phi(x)$.

NB As an aside, we can ask what is the relation between two conjugates of the same function, say $F_\phi = \phi^{-1} \circ f \circ \phi$ and $F_\psi = \psi^{-1} \circ f \circ \psi$. The answer is

Proposition 0.8 *$F_\phi = F_\psi$ iff $f \circ \theta = \theta \circ f$, where $\theta = \phi \circ \psi^{-1}$.*

Proof: This follows from the chain of equivalences

$$\begin{array}{l} F_\phi = F_\psi \\ \hline \phi^{-1} \circ f \circ \phi = \psi^{-1} \circ f \circ \psi \\ \hline f \circ \phi = \phi \circ \psi^{-1} \circ f \circ \psi \\ \hline f \circ \phi \circ \psi^{-1} = \phi \circ \psi^{-1} \circ f \\ \hline f \circ \theta = \theta \circ f \end{array}$$

In the particular case that $f = -id$, $F_\phi = F_\psi$ iff θ is an odd function.

Returning to theorem 0.1, we can prove the second part from corollary 0.6 and lemma 0.7:

Corollary 0.9 *If n is even, the only solutions of Babbage equation $f^n = id$ are id and the conjugates of $-id$.*

Proof: Any even number can be written as $n = 2^k \times odd$ with $k \geq 1$. The proof proceeds by induction on k.

Let $k = 1$. Assume that f is a solution of $f^{2\times odd} = id$. Letting $g = f^{odd}$, we have $g^2 = id$ whose only solutions are id and the conjugates of $-id$ (Lemma 0.7). Assume $g = id$. Then $f^{odd} = id$ and, by corollary 0.6, the only solution of this equation is $f = id$. If $\phi^{-1} \circ g \circ \phi = -id$, i.e., $\phi^{-1} \circ f^{odd} \circ \phi = -id$, we can re-write this equation as $(-\phi^{-1} \circ f \circ \phi)^{odd} = id$. Thus, by corollary 0.6 again, $(-\phi^{-1} \circ f \circ \phi) = id$. Equivalently, $f = \phi^{-1} \circ (-id) \circ \phi$. I.e., f is a conjugate of $-id$.

Assume that the result is true for k and prove it for $k+1$. Suppose that f is a solution of $f^{2^{(k+1)}\times odd} = id$ and let $g = f^{2^k \times odd}$. Then $g^2 = id$ and the only solutions of g are id and the conjugates of $-id$.

In the first case, $f^{2^k \times odd} = id$ and by induction hypothesis, the only solutions are id and the conjugates of $-id$.

In the second, $f^{2^k \times odd}$ is a conjugate of $-id$, i.e., there is a bijection ϕ such that $f^{2^k \times odd} = \phi^{-1} \circ (-id) \circ \phi$. Equivalently, $(\phi \circ f^{2^k \times odd} \circ \phi^{-1}) = -id$. But $\phi \circ (f^{2^k \times odd}) \circ \phi^{-1} = (\phi \circ f \circ \phi^{-1})^{2^k \times odd} = -id$ so that $h = (\phi \circ f \circ \phi^{-1})$ satisfies $h^{even} = -id$ which is impossible by proposition 0.4.

This concludes the proof of theorem 0.1.

As a corollary, we may find all the solutions (again continuous everywhere defined) of the dual Babbage functional equation

$$(^{**})\ \ f^n = -id$$

In fact,

Corollary 0.10 *If n is even $(^{**})$ has no solutions. If n is odd, the only solution of $(^{**})$ is $-id$*

Proof: The first part was proved above (Proposition 0.4). Assume n odd. From $f^n = -id$ we deduce that $f^{2n} = id$, and hence, from Theorem 1 either $f = id$ in which case $f^n = id$, contradicting $(^{**})$, or f is a conjugate of $-id$, i.e., there is an everywhere defined continuous bijection ψ such that $f = \psi^{-1} \circ (-id) \circ \psi$. Equivalently, for every x, $f(x) = \psi^{-1}(-\psi(x))$. We re-write this equation as

$$^{*}\ \ \psi(f(x) = -\psi(x)$$

On the other hand, $f = \psi^{-1} \circ (-id) \circ \psi$ implies that $f^n = \psi^{-1} \circ (-id)^n \circ \psi$. Since n is odd, $f^n = \psi^{-1} \circ (-id) \circ \psi$, i.e., $-x = \psi^{-1}(-\psi(x))$. This can be rewritten as

$$^{**}\ \ \psi(-x) = -\psi(x)$$

Combining * and **, $\psi(f(x)) = \psi(-x)$. Since ψ is a bijection, $f(x) = -x$.

NB Notice that ** is an immediate consequence of proposition 0.8. Indeed, F_{id} and $F_\psi = f$ are conjugates of $-id$. Therefore, $\theta = \psi \circ id^{-1} = \psi$ is an odd function.

0.1 Appendix

Theorem 0.11 *The only monotonically increasing everywhere defined continuous function solution of Babbage equation $f^2 = id$ is the identity function.*

Proof: Define the binary relation

$$R(t, s) \equiv f(1/2(t - s)) = 1/2(t + s)$$

Notice that by the property of f we could also write

$$R(t, s) \equiv f(1/2(t + s)) = 1/2(t - s)$$

We claim that if f is monotonically increasing, then R is functional, i.e., $R(t, s_1) \wedge R(t, s_2) \longrightarrow s_1 = s_2$.

Indeed, let s_1 and s_2 such that $R(t, s_1) \wedge R(t, s_2)$. Then either $s_1 < s_2$ or $s_1 = s_2$ or $s_1 > s_2$. Assume the first alternative, the last one is similar. Then $x_1 = 1/2(t - s_1) > 1/2(t - s_2) = x_2$. On the other hand $f(x_1) = 1/2(t + s_1) < 1/2(t + s_2) = f(x_2)$ contradicting the fact that f is monotonically increasing.

Notice that

$$(^*) \quad y = f(x) \quad \text{iff} \quad R(y + x, y - x)$$

Since $y = f(x)$ iff $x = f(y)$ (from Lemma 0.2 and the fact that f is its own inverse in this case),

$$(^{**}) \quad x = f(y) \quad \text{iff} \quad R(y + x, y - x)$$

From $(^*)$, it follows that

$$(^{***}) \quad R(f(x) + x, f(x) - x)$$

and from $(^{**})$,

$$R(y + f(y), y - f(y))$$

Replacing the dummy variable y by x,

$$(^{****}) \quad R(x + f(x), x - f(x))$$

From $(^{***})$ and $(^{****})$ and the functionality of R, $x = f(x)$ for all x.

References

[1] Lévy-Leblond, One more derivation of the Lorentz transformation, American Journal of Physics, vol 44, No 3, March 1976, 271-277

[2] J.F.Ritt, On certain real solutions of Babbage's functional equation, Annals of Mathematics, Second Series, vol 17, No 3, 1916, 113-122

[3] A.Royer, II Lorentz transformations from neither of Einstein's two postulates. Unpublished manuscript.