Full Einstein Field Equations

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16 April 2012

The Einstein field equations in the vacuum is

$$Ric = 0$$

where Ric is the Ricci tensor. In the usual tensor notation,

$$R_{ij} = 0$$

As shown in [1], to "deduce" this equation requires a 4-dimensional manifold endowed with a connection (or parallel transport) only, rather than a metric.

This is not so for the full Einstein field equations

$$R_{ij} - 1/2 g_{ij} R = -8\pi G T_{ij}$$

since the metric g_{ij} appears in the very statement of the equation.

The following note is a "deduction" of this full field equation.

What Einstein first thought was to find the exact analogue of Poisson equation $\nabla^2 \phi = 4\pi G \rho$ by writing

$$R_{ij} = C T_{ij}$$

where T_{ij} is the matter-energy tensor, since R_{ij}, the Ricci tensor was the analogue of the Laplacian. To check that he was in the right track, he computed the divergence of this tensor. From the conservation property of the matter-energy tensor T_{ij}, its divergence is 0. So if this equation were promising, the divergence of R_{ij} should also be 0. But it turned out that the divergence of R_{ij} was different from 0! Let us see why:

Computing the divergence of the Ricci tensor

A double contraction of the Bianchi identity

$$\nabla_q R^\gamma_{jlk} + \nabla_k R^\gamma_{jql} + \nabla_l R^\gamma_{jkq} = 0$$

leads to the sought result.

In fact, using the Einstein's convention and letting $\gamma = k$.

$$\nabla_q R^k_{jlk} + \nabla_l R^k_{jkq} + \nabla_k R^k_{jql} = 0$$

i.e.,

$$-\nabla_q R_{jl} + \nabla_l R_{jq} + \nabla_k R^k_{jql} = 0$$

Let $j = l$.

$$-\nabla_q R_{ll} + \nabla_l R_{lq} + \nabla_k R^k_{lql} = 0$$

I.e.,

$$-\nabla_q R + \nabla_l R^l_q + \nabla_k R^k_q = 0$$

Thus,

$$\nabla_l R^l_q = 1/2 \nabla_q R = 1/2 \delta^l_q \nabla_l(R)$$

But $g^l_q = g^{li} g_{iq} = \delta^l_q$. Hence,

$$\nabla_l R^l_q = 1/2 \nabla_q R = 1/2 g^l_q \nabla_l(R)$$

Using Ricci's theorem ($\nabla g = 0$)

$$\nabla_l(R^l_q) = \nabla_l(1/2 g^l_q R)$$

It follows that the symmetric tensor

$$G_{lq} = R_{lq} - 1/2 g_{lq} R$$

satisfies

$$\nabla_l G^l_q = 0$$

or, raising the index q (by using Ricci theorem),

$$\nabla_l G^{lq} = 0$$

i.e. its divergence is 0.

G_{lq} is called the *Einstein tensor*.

So, the Einstein field equation should be of the form

$$R_{\mu\nu} - 1/2 g_{\mu\nu} R = K T_{\mu\nu}$$

where K is a constant to be determined.

Newtonian limit

Assume that a particle is moving slowly in a weak stationary gravitational field. Recall that the equation of motion is

$$d^2x^\lambda/d\tau^2 + \Gamma^\lambda_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau) = 0$$

Since the particle is moving slowly, its velocity is negligible compared to the velocity of light ($c = 1$). Mathematically, we interpret negligible quantities as being nilpotents (or infinitesimals) and write $dx^\mu/d\tau \approx 0$. Thus all the terms in the sum other than the coefficient of Γ^μ_{00} are infinitesimals and the equation of motion reduces to

$$d^2x^\mu/d\tau^2 + \Gamma^\mu_{00}(dx^4/d\tau)^2 \approx 0$$

From now on, we let $t = x^4$ (i.e. 4 is a superindex).

Since the field is stationary, all time derivatives of $g_{\mu\nu}$ are infinitesimals and so,

$$\Gamma^\mu_{00} \approx -1/2g^{\mu\nu}\partial g_{00}/\partial x^\nu$$

(See relevant pamphlet ??.)

Since the field is week,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

with $h_{\alpha\beta} \approx 0$

Therefore,

$$\begin{aligned}
\Gamma^\mu_{00} &\approx -1/2g^{\mu\nu}\partial g_{00}/\partial x^\nu \\
&\approx 1/2(\eta^{\mu\nu} + h^{\mu\nu})(\partial\eta_{00}/\partial x^\nu + \partial h_{00}/\partial x^\nu) \\
&\approx -1/2(\eta^{\mu\nu}\partial h_{00}/\partial x^\nu + h^{\mu\nu}\partial h_{00}/\partial x^\nu) \\
&\approx -1/2g^{\mu\nu}\partial h_{00}/\partial x^\nu \\
&\approx -1/2\nabla h_{00}
\end{aligned}$$

The equation of motion becomes

$$d^2x/d\tau^2 \approx 1/2(dt/d\tau)^2\nabla h_{00}, \qquad d^2t/d\tau^2 \approx 0$$

Dividing the first “equation” by $(dt/d\tau)^2$

$$d^2x/dt^2 \approx 1/2\nabla h_{00}$$

Comparing this equation with Newton's

$$d^2x/dt^2 = -\nabla\phi$$

and we conclude that

$$h_{00} \approx -2\phi + C$$

where C is a constant. By choosing the potential to be 0 at infinity C=0. Thus $g_{00} \approx -(1+2\phi)$ On the other hand, by definition of the energy-density tensor (see Wiki: "Stress-energy tensor").

$$T_{00} \approx \rho$$

But we have Poisson equation

$$\nabla^2\phi = 4\pi G\rho$$

Furthermore

$$g_{00} \approx -(1+2\phi)$$

and

$$T_{00} \approx \rho$$

From all of this we conclude that

$$\nabla^2 g_{00} \approx -8\pi G T_{00}$$

and $K \approx -8\pi G$. To obtain the final Einstein's equation

$$R_{\mu\nu} - 1/2 g_{\mu\nu} R = -8\pi G T_{\mu\nu}$$

we identify K with its approximate value, since both should be real numbers (Postulate of Physics)

Uniqueness of Einstein tensor?

Notice that $G_{\mu\nu} = R_{\mu\nu} - 1/2 g_{\mu\nu} R$ has the following properties:

1. The quantities $G_{\lambda\mu}$ depend only on the gravitational potentials and their first and second order derivatives. Furthermore, they are linear with respect to the second order derivatives

2. The quantities $G_{\lambda\mu}$ satisfy the conservation equations

$$\nabla_\mu G^{\lambda\mu} = 0$$

Elie Cartan has shown that the only tensors $S_{\lambda\mu}$ satisfying the above conditions are given by

$$S_{\lambda\mu} = h[R_{\lambda\mu} - 1/2 g_{\lambda\mu}(R + k))]$$

where h and k are constants. Suppressing h, we obtain the most general Einstein field equation

$$R_{\mu\nu} - 1/2 g_{\mu\nu}(R + k) = -8\pi G T_{\mu\nu}$$

Check Weinberg's book [2] where a proof is presented.

References

[1] G.E.Reyes, A derivation of Einstein's Vacuum Field Equations, *Models, Logic and Higher Dimensional Categories*, CRM Proceedings and Lecture Notes 2011

[2] Steven Weinberg, *Gravitation and cosmology: principles and applications of the General Theory of Relativity*, John Wiley and Sons 1972