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REVIEW ARTICLE

# Topos Theory in Montréal in the 1970s: My Personal Involvement

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*A la memoria de Emilio del Solar Petit por mostrarme los caminos de la lógica y de la música en los remotos tiempos de nuestra juventud*

As the title suggests, this paper will do no more than tell a story of a few developments in topos theory in which I was personally involved during the 1970s at Montréal. Mind you: a story and not a history! I would like to give a personal account of the times (as I lived them), of the heroes (as I knew them), of the hopes raised and foundered and of the dominant ideology present on the everyday work (as I felt it).

The subjects that I will discuss are: logics and their doctrines, completeness theorems, generic models and classifying toposes and early work on Synthetic Differential Geometry.

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## 1. Logics and their doctrines

I came to Montréal in 1967 when 'La Révolution tranquille' was quietly bringing out 'La Belle Province' into the light from the depths of 'La Grande Noirceur'[2] That was the year of the official opening of Montréal to the world through Expo 67, and the beginning of the Trudeau era that was to change Canada enormously. I had come with a fresh PhD from Berkeley in Logic with a thesis on model theory, written under the direction of William Craig. One of my first mathematical contacts was Jean Maranda who told me that he knew only one thing about model theory: it was the study of a couple of adjoint functors between theories and models. These were news to me! Obviously in Montréal, the air that you breathed was different from that of Berkeley where categories were not even once mentioned during my six years as graduate student.

My first significant mathematical contact in my new city that was to prove determinant for a great part of my career was André Joyal. At the time of my arrival he was a student at Montréal University, working his way through Grothendieck's *Eléments de Géométrie*

Proof Only

[1] This paper is based on a talk that I gave at a meeting of the 'Atelier d'Histoire des Catégories' in Montréal University on 13 September 2001.

[2] 'La Grande Noirceur' (The Great Darkness), refers to the autocratic, corrupted and reactionary regime of Maurice Duplessis, the Premier of Québec ('La Belle Province') from 1936 to 1939 and again from 1944 to 1959. Duplessis had the active support of the Catholic Church. 'La Révolution Tranquille' (The Quiet Revolution) refers to the intense socio-political and socio-cultural change in Québec brought about by the liberal government of Jean Lesage in 1960 and continued by his successors. The government was secularized and took direct control of health care and education which had been in the hands of the Catholic Church.

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*Algébrique* (*Grothendieck and Dieudonné 1960*) and several volumes of the *Séminaire de Géométrie Algébrique du Bois Marie* (*Grothendieck 1972*). I believe that this was the site where he learned category theory and toposes. I learned category theory mostly from him and I imagine that he learned some model theory from me. During these earlier years I studied Lawvere's work on logic and categories (*Lawvere 1963*, *1965*, *1969*, *1970*), which culminated in the creation of elementary topos in joint work with Tierney in 69–70 (*Tierney 1973*) during a very active year at Dalhousie University just before he was expelled from that university because of his 'un-canadian' political activities. I believe it is important to point this out, since his leninist-marxist's ideology was to play a role in the development of categorical logic.

I also learned about Lawvere's (*1963*) and Bénabou's (*1968*) ideas to do Universal Algebra categorically. And by 1970/1971 Joyal and I were already trying to extend their ideas to do first-order logic categorically, following a basic discovery of Lawvere that started categorical logic, namely that the (categorical counterparts of) existential and universal quantifiers are left and right adjoints to pull backs (the counterpart of substitution). These are called images and dual-images respectively (see e.g. *Lawvere 1970*).

Recall that if $f : X \longrightarrow Y$ is a function between sets and $B$ is a subset of $Y$, then the pull-back of $B$ along $f$ is the subset of $X$ defined by $f^*(B) = \{x \in X : f(x) \in B\}$. By considering $B$ as a predicate of $Y$, say $B(y)$, the pull-back may be considered as the predicate of $X$ obtained from $B(y)$ by substituting $f(x)$ for $y$. Lawvere's remark was that the pull-back formation

$$f^* : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$$

which is a functor between posets has a left and a right adjoints

$$\exists_f \dashv f^* \dashv \forall_f$$

Indeed, by defining

$$\begin{cases} \exists_f(A) = \{y \in B : \exists x(f(x) = y \text{ and } x \in A)\} \\ \forall_f(A) = \{y \in B : \forall x(f(x) = y \text{ implies } x \in A)\} \end{cases}$$

it is easy to check that

$$\exists_f(A) \subseteq B \; iff \; A \subseteq f^*(B)$$

which is the meaning that $\exists_f \dashv f^*$. Similarly, one can check that $f^* \dashv \forall_f$, i.e.

$$f^*B \subseteq A \; iff \; B \subseteq \forall_f(A)$$

Lawvere's ideas motivated Volger (*1971*) to do first-order logic categorically, but his attempt was only partially successful. Although we partly followed his lead, we realized that Volger's 'mistake' was to incorporate $\Omega$, the subobject classifier in the category associated to a theory, an object which does not belong to first-order logic.[3]

NB Almost, if not all the notions used in this paper are defined in *Mac Lane and Moerdijk 1992*. The first pages of this book describe the categorical preliminaries required to understand the material of this paper.

One advantage of making categories out of logic that proved to be essential for the development of Synthetic Differential Geometry later on was the fact that one could define

[3] In set theory $\{0, 1\}$ classifies subsets by means of characteristic functions in the sense that if $Y \subset X$, then the characteristic function $\xi_Y : Y \longrightarrow \{0, 1\}$ is 1 for $x \in Y$ and 0 for $x \notin Y$. This was generalized to toposes by Lawvere with $\Omega$ playing the role of $\{0, 1\}$ (see e.g. *Johnstone 1977*).

Proof Only

models of a theory in categories other than *Sets*, in strict parallel with categorical Universal Algebra, where the notions of Group, algebra, etc, *in a category* were available from the work of Lawvere and Bénabou, already mentioned. Thus, a topological group is a group object in the category of topological spaces, a Lie group is just a group in the category of manifolds, etc.

The following table sums up the basic idea of categorization of logic:

| *logic* | *category theory* |
|---|---|
| (many-sorted) theory | category |
| sort | object (of a category) |
| (sorted) formula | sub-object |
| (sorted) term | morphism |
| interpretation | functor |
| set-theoretical model | functor into *Sets* |
| homomorphism | natural transformation |

In this table, category refers to categories with finite limits (pull-backs and final object) with possibly further structure satisfying some exactness conditions. This extra structure is determined by the choice of the logical connectives considered.

In the process of our studies, we were able to discover several categorical counterparts (or 'doctrines' as they were called later (*Kock and Reyes 1977*)) of theories formulated using some of the logical connectives. The most important one was the doctrine of coherent categories corresponding to coherent theories. A coherent theory is a set of sequents of the form $\phi \vdash \psi$, where $\phi$ and $\psi$ are coherent formulas, i.e. formulas written in the fragment corresponding to the choice of the logical connectives $\wedge, \top, \vee, \bot$ and $\exists$[4]. A coherent category is one with finite limits, finite sups of subobjects of a given object and images. An image is an epimorphism $f : A \longrightarrow B$ such that $f$ does not factor thru a proper subobject of $B$ (cf. *Makkai and Reyes 1977*, p. 66). The exactness conditions are simply that sups and images should be stable under pull-backs.

The passage from a coherent theory to its categorical counterpart, namely a coherent category, goes roughly like this: objects are coherent formulas and morphisms are (equivalent classes of) coherent formulas which are provably (in the sense of the theory) functional relations. The actual details proved to be rather long and were worked out by my student J. Dionne (*1973*) in his Master thesis at Montréal University in 1973. Another important doctrine was that of Heyting categories corresponding to intuitionistic first-order theories. A Heyting category is a coherent category such that pull-backs have right adjoints. Grothendieck's toposes as well as elementary toposes are examples of Heyting categories.

Coherent logic (in the many-sorted version) and its infinitary generalization, geometric logic, seem to have found their way in the 'folklore' of topos theorists and some schools of computer scientists.

One of the gratifying experiences brought by our work was that some of the categorical structures obtained were in fact already known in other domains. This was in a marked contrast with Algebraic Logic. As an important example, a particular case of a coherent category, namely the notion of a pretopos, was already studied by the Grothendieck school. Roughly speaking a pretopos is a coherent category with finite co-products and quotients

[4] Strictly speaking, sequents should have a context and thus should be written as $\phi(x) \vdash_X \psi(x)$, where $X$ is a sequence of variables, but although important, we will omit this complication here.

Proof Only

by equivalent relations. Furthemore, a coherent category generates freely a pretopos. Thus we can define the pretopos $\mathbb{P}(T)$ associated to a theory $T$, as well as the coherent category $\mathbb{C}(T)$ associated to the theory. Furthermore, the set theoretical models of the theory $T$ are the same as the pretopos morphisms of $\mathbb{P}(T)$ into the category of sets. This will be important in the discussion of the conceptual completeness theorem.

The construction of a category from a theory was the exact analogue of the Lindenbaum-Tarski Boolean algebra associated to a propositional theory and it was natural to ask whether every coherent category comes from a coherent theory. I believe that this was first published in my book with *Makkai and Reyes 1977*. For an outlook of some aspects of this work see *Reyes 1977*. Since then, several other logics have been categorized, including type theory. The corresponding doctrine for this last one turns out to be precisely the elementary toposes of Lawvere/Tierney. This was proved independently by *Boileau and Joyal. 1981*, *Coste 1974* and *Fourman 1977* (although his formalization was rather unusual).

With these ideas in place, several notions were readily available, even some that eluded logicians: the interpretation of one theory into another, the difference between presentation of a theory (which is what ordinary logic is about) and the theory itself (that could not be represented in ordinary logic).[5] Another problem that worried us at the time was the fact that concepts were not adequately dealt with in ordinary logic. Take for instance the concept of 'unit' in (commutative) ring theory. Clearly this concept is not a formula, since the same concept could be expressed equally well by $\exists y(xy = 1)$ or $\exists z(zx = 1)$ or even $\exists! y(yx = 1)$. Of course this suggests that a concept should be an equivalence class of formulas, where $\phi(x)$ is equivalent to $\psi(x)$ provided that the sequents $\phi(x) \vdash \psi(x)$ and $\psi(x) \vdash \phi(x)$ are provable in the theory $T$. The development of these ideas led to Algebraic Logic in either of its two forms: Halmos' polyadic algebras and Tarski's cylindric algebras, which are both Boolean algebras with operations.

Nevertheless, Algebraic Logic met with some difficulties. From our point of view, the main problem is that the resulting structures, Boolean Algebras with operators, are ad-hoc structures which are foreign to those studied by mathematicians. This is not so for categories coming from logic, i.e. toposes that had previously studied by the Grothendieck school. As an example, Gödel completeness theorem had been proved by Deligne in this context (*1972*). Several other similar results had also been proved.

The 'logical' categories appear as categories of concepts and Categorical Logic itself appears then as the 'missing link' between signs and sets. We thus have three levels

| Linguistic level | |
|---|---|
| Conceptual level | categories |
| Semantical level | models |

## 2. Completeness theorems

Model theory tells us what a set-theoretical model of a theory is. What is the corresponding notion of model of a category in Categorical Logic? As mentioned already (see the first table) it should be just a functor from the category into *Sets* which preserves this extra structure. In particular, a model of a coherent category is a functor from the category into *Sets* which preserves finite limits, finite sups (of subobjects) and images. Such

[5] I recall that Tarski told us in his course on 'General Algebraic Systems' that there was no such a thing as *the* theory of groups, but rather an infinite number of theories of groups, according to the choice of primitive operations and axioms!

Proof Only

a functor is called a coherent functor. And indeed, we have a bi-univocal correspondence between the models of a coherent theory (in the model-theoretical sense) and the coherent functors from the (coherent) category associated with the theory.

Since every coherent category $\mathbb{T}$ is obtained from a coherent theory, Gödel's theorem gives a completeness theorem for coherent categories.

By putting together all the set-theoretical models of the category one obtains an embedding

$$e : \mathbb{T} \longrightarrow Sets^{|Mod(\mathbb{T})|}$$

into the category of families of sets indexed by the models of the theory, namely the evaluation defined on objects by $e(A)(M) = M(A)$. (The definition of $e$ on morphisms should be clear)[6]

Using these techniques, one can prove several completeness theorems, by just transferring those proved in logic. Let us give two examples:

Kripke's completeness theorem for an intuitionistic theory (*Kripke* 1965) shows that any Heyting category $\mathbb{H}$ has an embedding

$$e : \mathbb{H} \longrightarrow Sets^{\mathbb{P}}$$

into the category of set-valued functors from the dual of a poset $\mathbb{P}$ (on which further conditions could be imposed).

On the other hand, Mansfield's completeness theorem for geometric theories (*Mansfield* 1972) (i.e. whose sequents are of the form $\phi \vdash \psi$ with $\phi$ and $\psi$ formulas in the language whose connectives are $\wedge, \top, \bigvee, \bot, \exists$) shows that every geometric theory $\mathbb{T}$ has an embedding

$$e : \mathbb{T} \longrightarrow V^{(\mathbb{B})}$$

in the category of Boolean-valued sets of Scott–Solovay (with $\mathbb{B}$ a complete Boolean algebra) (*Bell* 1977).

Now, all of these 'target' categories are Grothendieck topos and the point is that these embeddings preserve the corresponding structure of the 'source' categories. Thus, the first is coherent, the second is Heyting and the last is geometric (i.e. it is coherent and preserves arbitrary $\bigvee$). Furthermore, it should be clear that these embeddings are *models* of the corresponding theories. But how do we interpret formulas and sequents directly in a topos? For these particular toposes, the answer had been given by Kripke and Cohen, but a general solution to this problem was missing. It is interesting to notice that the Grothendieck school was confronted with this problem when trying to define structures like 'local ring' or 'separably real-closed local rings' in a topos. Their method was to use points. Thus a ring $R$ was local provided that for every point $p$, $p^*(R)$ is a local rings in Sets. There were problems with this solution, however. Some toposes had no points at all. Even worse, for some structures, this solution was not correct. In fact, it was OK for structures defined in coherent logic only. The correct solution was obtained by Joyal using a generalization of the 'forcing' of Kripke. Since then, this general notion is called the Kripke–Joyal forcing (*Johnstone* 1977).

A few words about the so-called 'internal language' of a topos (*Mac Lane and Moerdijk* 1992). Quite correctly, this tool is attributed today to Bénabou and Mitchell. But how come that this tool was not discovered earlier, after the categorization of logic achieved already and the fact that Joyal had already shown how to interpret formulas and sequents in a topos?

[6] $|Mod(\mathbb{T})|$ is in general a proper class, not a set, but I will pass over this difficulty that may be overcome in several ways

Proof Only

The answer, I believe, should be found in our interpretation of the dominant ideology of the period at least on this side of the mountain[7] : dialectical materialism as understood by *Mao 1971* and explained to us by Lawvere, with its emphasis on the universality of contradictions, main contradictions, the dominant and secondary aspects of a contradiction and so on.[8] According to Lawvere, the main contradiction in the theory of topos was that between Geometry and Logic. Indeed, a topos has two aspects: the geometric and the logical one. From the geometrical point of view, this theory was the study of concepts of local character (sheaves) that appeared in Topology, Algebraic Geometry, Analytic Geometry. From the logical point of view, topos theory provided a flexible language to describe the various types of 'variable sets' that had appeared in the independence results of set theory, in infinitary logics and in non-standard analysis. Thus, topos theory was the theory of 'variable sets'. Of these aspects, the geometric one was dominant, while the logical aspect was secondary.

Some people misunderstood these 'contradictions' and read 'reactionary' for 'secondary' and 'progressive' for 'dominant'. From this misunderstanding, the idea came up rather naturally: why not eliminate the logical aspect altogether? This was not in Lawvere's mind, but was natural enough after what had been accomplished. Didn't the work done first by Bénabou and Lawvere and then by us in Montréal showed precisely that the logico/linguistic aspect of logic could be avoided, thus proving that logic could be 'geometrized'? So, the language was there all along and we could have used it without problems, but it was a mistake to use it. In fact, it was more than a mistake: it was reactionary or fascist (to employ a word freely used in that period)[9]

It took some time to achieve what it seems to me a balanced view of the role of logic and language in topos theory: namely that notions may best be defined categorically (just as in Differential Geometry, notions may be best defined in 'invariant form') but logic may be used freely to prove things about these (just as in Differential Geometry the use of coordinates may simplify calculations or even make them possible).

Returning now to our embeddings: all of these give models of the category in question into a topos. The question was whether there were categorical formulations and proofs of at least some of these results and here Joyal proved a fundamental theorem to the effect that if $\mathbb{T}$ is a coherent category, the evaluation functor

$$e : \mathbb{T} \longrightarrow Sets^{Mod(\mathbb{T})}$$

is a conservative coherent functor which preserves those dual-images (i.e. rights adjoints to pull-backs) which happen to exist. This theorem is an invariant version of Kripke's theorem and shows the central role of coherent logic in intuitionistic logic (*Makkai and Reyes 1977*).

One of the important consequences of the completeness theorems for coherent logic was the following 'meta-theorem' that Makkai and myself proved in the book already mentioned: to prove a sequent of coherent formulas of the form $\phi \vdash \psi$ in a topos, it is enough to prove it in *Sets*.

An example is a new proof of Swan Theorem on vector bundles (Reyes 1978).

7 The universities of Montreal and McGill are separated by a mountain: Mont Royal.

8 This terminology was quite unusual for most people at the time. I remember that Lawvere asked in one of his talks 'What is the main contradiction of Arithmetic?' and Jim Lambek answered 'Gentzen proved that Arithmetic is free of contradictions'.

9 As a personal aside, I recall that a group of Italians who attended the 1974 meeting in Montréal where these results were presented for the first time carried out a mass democracy against Barbara Veit Riccioli for showing an interest in my 'reactionary' talks whose aim was to use logic to prove some results in topos theory (In the terminology of the period, a 'mass democracy' was a collective accusation against a 'reactionary' point of view of somebody)

proof only

## 3. Generic models and classifying toposes

As mentioned already, one of the consequences of categorizing theories is the possibility of defining models in universes other than *Sets*. This possibility of changing universes is at the base of the notion of *generic* model. To understand this notion, let us turn to a key theorem of Kronecker in the theory of rings.[10] Up until the middle of the nineteenth century, even people like Gauss, Galois and Abel took for granted that any polynomial with coefficients in a ring $A$ had a zero *somewhere*, in an extension of $A$, although this had not been proved (beyond those cases covered by the Fundamental Theorem of Algebra).

Kronecker, following the lead of Cauchy, made the following basic observation: let $A$ be a commutative ring and $p(X)$ a non-constant polynomial with coefficients in $A$. Although $p(X)$ may fail to have a root in $A$, there is an extension of $A$ which contains a root of $p$. Let us take the example due to Cauchy. Let $\mathbb{R}$ be the real numbers and let $p(X) = X^2 + 1$. Clearly, $p$ does not have any zeros in $\mathbb{R}$. But look at the ring $\mathbb{R}[X]/(p)$. Here $p(X)$ has a zero, namely $G = X + (p)$. In fact, $p(G) = G^2 + 1 = (X + (p))^2 + (1 + (p)) = (X^2 + 1) + (p) = (X^2 + 1)\ mod(X^2 + 1) = 0$.

Returning to the general case, Kronecker pointed out that the ring $A[X]/(p)$ where $p$ is any polynomial contains the 'generic' zero $G = X + (p)$. Furthermore, this solution is universal in the sense that (taking $A = \mathbb{Z}$ to simplify), $G^* : RING(\mathbb{Z}[X]/(p), R) \simeq Zero_R(p)$, where $RING(\mathbb{Z}[X]/(p), R)$ is the set of ring morphisms from the ring $\mathbb{Z}[X]/(p)$ to an arbitrary ring $R$, $\phi$ one of them and $G^*(\phi)$ is the obvious zero of $p$ in $R$ obtained from $\phi$, namely $\phi(G)$. Clearly, all of this can be generalized to ideals of polynomials, rather than single polynomials.

This result may be interpreted as showing that although the 'domain' $A$ may fail to have a zero of a proper ideal $I \subset A[X_1, X_2, \ldots, X_n]$ there is an extension of this 'domain' namely $A[X_1, X_2, \ldots, X_n]/I$ which results from $A$ by adding the generic zero $G$ of $I$.

What does all of this have to do with logic? The point is that there is a far reaching analogy, stressed by Joyal, between rings and toposes:

| *ring theory* | *categorical logic* |
|---|---|
| ring | topos |
| finitely presented ring | coherent topos |
| $\mathbb{Z}$ | Sets |
| ideal | theory |
| zero | model |
| proper ideal | consistent theory |
| generic zero | generic model |

What should a generic model be? The classifying topos provides an answer.

Of all the results that Joyal and I obtained in the 1970s, the most important and most widely quoted was the existence of the classifying topos of an arbitrary coherent theory. It was discovered as an attempt to 'solve the main contradiction between Logic and Geometry' in topos theory that Lawvere had formulated as the search of certain adjoint functors (*Reyes 1974*). For a better and more complete exposition, see *Makkai and Reyes 1977*.

If $T$ is a coherent theory, there is a topos $\mathbb{B}(T)$ and a 'generic' model $G$ of $T$ in $\mathbb{B}(T)$ satisfying the universal property: $G^* : TOP(\mathbb{E}, \mathbb{B}(T)) \simeq MOD_{\mathbb{E}}(T)$. where $G^*$ is the functor which sends a geometric morphism $(p^*, p_*)$ into the model $p^* \circ G$ of $T$ in $\mathbb{E}$. (We recall

[10] In fact, Kronecker proved this theorem for fields, but for our purposes rings are enough.

Proof Only

that a geometric morphism $p : \mathbb{F} \longrightarrow \mathbb{E}$ is a couple of adjoint functors $p^* \dashv p_*$ such that $p^*$ preserves finite limits.)

The topos $\mathbb{B}(T)$ is called the *classifying topos* of the theory $T$ and it is a coherent topos in the sense of Grothendieck.[11] The analogy with the Kronecker's construction should be clear.

On the basis of this analogy we may view the classifying topos of $T$ as the 'universe' $\mathbb{B}(T)$ which results from *Sets* by forcing the existence of a generic model of $T$.

It turns out that every coherent topos is the classifying topos of a coherent theory and it is interesting to find this theory for well-known examples. The Zariski topos, for instance, is the classifying topos of the theory of local rings. To simplify, we shall say that the Zariski topos classifies local rings.

Remember that the theory of local rings may be axiomatized by the following coherent theory $T$ in the language $L$ that has one sort, 'the underlying set of a ring' and whose non-logical symbols are the following operation symbols: 0,1 (0-ary) and $+, -, \cdot$ (binary) and the following axioms:

$$\begin{cases} \text{(i)} \;\; 0 = 1 \vdash \bot \\ \text{(ii)} \;\; \top \vdash \; \forall x[\exists y(x \cdot y = 1) \vee \exists y((1-x) \cdot y = 1)] \end{cases}$$

On the other hand, the topos of simplicial sets classifies linear orders with bottom and top elements which are different. This example is due to Joyal. Other examples have appeared naturally in several investigations.

I will sketch a proof that the classifying topos of the theory $T$ of (non-trivial) local rings is the Zariski topos. This particular example of classifying topos is due to *Hakim 1972*, but the proof is the one in *Makkai and Reyes 1977* (chapter 9), where further details can be found.

Let $\mathcal{R}_f$ be the category of finitely presented rings, i.e. quotients of polynomial rings $Z[x_1 \cdots x_n]$ by ideals generated by finitely many polynomials in $Z[x_1 \cdots x_n]$ ($Z$ is the ring of integers). Define $\mathcal{C}$ to be the opposite of $\mathcal{R}_f$. We make $\mathcal{C}$ into a site by introducing the Zariski topology on $\mathcal{C}$ as follows. First, notice that a covering family $(A_i \longrightarrow A)$ in $\mathcal{C}$ becomes a co-covering family $(A \longrightarrow A_i)$ in $\mathcal{R}_f$. The Zariski topology on $\mathcal{C}$ is generated by the following co-covering families

(i) the empty family co-covering the zero ring

(ii) $\{\mathrm{A} \longrightarrow \mathrm{A}[1/\mathrm{a}],\ \mathrm{A} \longrightarrow [1/\mathrm{b}]\}$ co-covering A

whenever $a+b=1$ and $A[1/a] = A[x]/(1-xa)$.

The Zariski topology on $\mathcal{C}$ can be generated by two co-covering families:

(i) as above and

(ii)' $\{\mathrm{Z[x]} \longrightarrow \mathrm{Z[x, 1/x]}, \mathrm{Z[x]} \longrightarrow \mathrm{Z[x, 1/(1-x)]}\}$

The Zariski topos $\mathcal{Z}$ is the topos whose site is $\mathcal{C}$. Let $\epsilon : \mathcal{C} \longrightarrow \mathcal{Z}$ be the obvious functor. We can interpret the language $L$ in the category $\mathcal{C}$ by $M : L \longrightarrow \mathcal{C}$ as follows: $s$ is the unique sort of $L$. Then $M(s) = Z[x]$. The terminal object 1 of $\mathcal{C}$ is the initial object $Z$ of $\mathcal{R}_f$. Accordingly, $M(0)$ is defined as the unique morphism $\alpha : Z[x] \longrightarrow Z$ such that $\alpha(x) = 0$. Similarly for $M(1)$.

[11] Without going into details, toposes defined by topologies having the property that every covering family is finite are coherent.

Proof Only

The product $Z[x] \times Z[x]$ in $\mathcal{C}$ is the coproduct $Z[x_1, x_2]$. Now, $M(+)$ is defined as the unique morphism

$$\alpha : Z[x] \longrightarrow Z[x_1, x_2]$$

such that $\alpha(x) = x_1 + x_2$. Similarly for $-$ and $\cdot$. One can show that the interpretation just defined satisfies all the axioms of a commutative rings with 1. The classifying topos $\mathbb{T}$ is the Zariski topos and the *generic model* of $T$ in the Zariski topos is the composite $G = \epsilon \circ M : L \longrightarrow \mathcal{Z}$.

An important tool to find the coherent theory classified by a coherent topos is the conceptual completeness of Makkai and myself (*Makkai and Reyes* 1977) which says, in one of the simplest formulations, that 'points are enough for classifying'. In more detail, if $P_1$ and $P_2$ are pretoposes and $I : P_1 \longrightarrow P_2$ is a pretopos morphism such that $I^* : Mod(P_2) \longrightarrow Mod(P_1)$ is an equivalence, then $I$ itself is an equivalence. (Here $Mod(P)$ is the set of set models of $P$.)

I consider this theorem and its variants as the most important result in our book. When it was still a conjecture, I proposed this problem to Mihály Makkai who had just arrived to McGill and whom I knew from Berkeley and his contribution to the solution of this problem, or should I say, his proof of this conjecture was the beginning of our collaboration that culminated in our book on first-order categorical logic, quoted several times already. The result in question is interesting for another reason: it shows that finite disjoint sums of formulas and quotients of a formula by a provable (in the theory) equivalence relation are coherent logical operations (although not representable in ordinary logic), but there are no others. It has been widely used and quoted in the literature and a purely categorical proof has been given by *Pitts* 1987.

One of the most interesting applications was the proof by Michel Coste and Marie-Françoise Roy that the real spectrum of a formally real ring (a topological space) contained the same information that the topos-theoretic real spectrum (i.e. a topos) (*Coste and Coste* 1980). Once that this was achieved, the Costes 'threw away the ladder after they climbed it', talked only about the topological space and achieved respectability in the French Mathematical Establishment that was never too kin, at least during this period, about category theory or logic anyway.

Nevertheless, several results proved by the Grothendieck school were in fact well-known in the context of model theory. As an example, Deligne's theorem that coherent toposes had enough points (*Deligne* 1972) turns out to be, modulo the theory of classifying toposes, equivalent to Gödel's completeness theorem for first-order logic. Apart from Deligne's proof there is another by Joyal, purely categorical, but with strong similarities to Henkin's proof of the completeness theorem. In our book with Makkai, we prove still another theorem on the existence of points for a *separable* Grothendieck topos:

A separable Grothendieck topos has enough points

We say that a topos is *separable* if it is equivalent to the category of sheaves over a separable site with finite limits. A separable site, in turn, is one whose underlying category is countable, i.e. it has countably many objects and $Hom(A, B)$ is countable for each $A, B \in ob(\mathcal{C})$ and, furthermore, the system of basic coverings of the site is countable.

Another result of Grothendieck about points of a topos being obtained as co-limits of a set of points is a consequence of the downward Lowenheim–Skolem–Tarski theorem.

Returning to classifying toposes: from the existence of the classifying topos and its 'converse' it follows that any coherent topos results from *Sets* by adding a generic model of a suitable coherent theory, in parallel to the fact that any finitely presented ring may be

Proof Only

obtained as a quotient of a ring of polynomials with coefficient in $\mathbb{Z}$ divided by a suitable ideal.

The classifying topos theorem has an infinitary extension, as we showed with Makkai in our book already quoted. In fact, it turns out that every geometric theory has also a classifying topos and, furthermore, that every topos is the classifying topos of a geometric theory. Thus, every Grothendieck topos appears as the extension of *Sets* obtained by adding a generic model of a suitable geometric theory, in analogy with the representation of an arbitrary ring as a ring of ploynomials (in a possibly infinite number of indeterminates) modulo some ideal.

I don't know to what extent this analogy 'rings/toposes' has been pursued by later workers.

Although classifying spaces were well-known in Topology at the time, classifying toposes for theories were new, although particular examples of classifying toposes had been described by Hakim (*1972*), a student of Grothendieck. Nevertheless, the logic was hidden.

Through the use of classifying toposes, completeness theorems could be formulated in our book with Makkai (loc.cit.) in terms of existence of geometric morphisms. For instance Mansfield theorem implies an improved version of Barr's theorem (*Barr 1974*) to the effect that every topos $\mathbb{E}$ has a surjective geometric morphism

$$p : Sh(\mathbb{B}) \longrightarrow \mathbb{E}$$

where $\mathbb{B}$ is a complete Boolean Algebra. (We say that $p$ is surjective if $p^* : \mathbb{E} \longrightarrow \mathbb{B}$ is faithful.)

This improved version was used later by Makkai and myself on our work on intuitionistic and modal logic in 1995 (*Makkai and Reyes 1995*).

Another theorem that became important later on was that every topos $\mathbb{E}$ with enough points (e.g. a coherent topos) has an open surjective geometric morphism

$$p : Sh(X) \longrightarrow \mathbb{E}$$

with $X$ a topological space 'of points'.

I should add that Makkai and I did not use this geometric terminology that was not available at that time, but stated this result in terms of the inverse image part of the geometric morphism $p$, namely $p^*$ preserves dual images. This theorem was improved by Carsten Butz (using ideas of Joyal and Moerdijk) in his PhD thesis (*Butz 1996*) where he proved that $p$ could be chosen furthermore connected, locally connected and in such a way that $p^*$ induces isomorphisms in cohomology. Thus, as he puts it 'even though toposes were invented to define cohomology theories which came from very different structures, many of these cohomologies can be realized as the cohomology of some topological space'. If truth should be told, however, one should say that these topological spaces are rather unappealing.

## 4. Early work on synthetic differential geometry

By the mid seventies I started to look for possible applications of topos theory and categorical logic to other domains of mathematics. There were of course the spectacular proofs of the Weil conjectures due to Grothendieck and Deligne, which used topos theory but my aim was far more modest. I was intrigued by the possibility of formulating theories that, although classically inconsistent, were perfectly OK. The prime example, it seemed to me, was Synthetic Differential Geometry (SDG).

Proof Only

This theory had been introduced by Lawvere in 1967 in talks given at the University of Chicago. The basic idea is to follow the intuition of some of the seventeenth century creators and precursors of calculus that 'in the infinitely small every curve is a line'. Lawvere's context was a cartesian closed category with a commutative ring with unit $R$ ('the line'). One can define the object $D$ of first-order infinitesimals by the equalizer:

$$D \xrightarrow{e} R \underset{(\ )^2}{\overset{O}{\rightrightarrows}} R$$

where $O$ is the composite $R \xrightarrow{!} 1 \xrightarrow{0} R$.

The main axiom is that the canonical map $\alpha : R \times R \longrightarrow R^D$ defined as the exponential adjoint of the composite

$$R \times R \times D \xrightarrow{R \times \pi} R \times R \xrightarrow{+} R$$

where $\pi$ is the restriction of the product map $R \times R \xrightarrow{\cdot} R$, is a bijection. Intuitively, $D$ are the reals of square 0 and $\alpha$ is the map that sends the couple $(a, b)$ to the infinitesimal line $[d \to a + db]$. Thus, any function $f : D \longrightarrow R$ is given uniquely by a line represented by the couple $(a, b)$ whose slope is $b$. This allows us to define the *derivative* of $f$ at 0 by putting $f'(0) = b$. Using translations one can define derivatives at other reals.

Before Lawvere's work, there had been some attempts along these lines. The most important were those of Ehresmann ('jets') (*Ehresmann 1953*), Weil ('points proches') (*Weil 1953*) which is an extension of the first, and Grothendieck (use of nilpotent elements in Algebraic Geometry (*Grothendieck and Dieudonné 1960*)). Nevertheless, the theory remained underdeveloped perhaps due to the lack of a direct, flexible language to handle these infinitesimal spaces. Ehresmann told me in 1975 that he introduced jets to give firm foundations to Fermat's ideas on Differential Calculus that preceded the work of Newton and Leibniz (*Ehresmann 1953*). Weil ideas were close to this program, but everything was dualized. This made hard to follow our geometric intuitions. For instance, instead of spaces he had rings of smooth functions and, instead of infinitesimals, finite dimensional local rings the prototype of which was the ring $C^\infty(\mathbb{R})/(x^2)$. There was no language to describe these spaces directly. In one of his visits to Montréal Weil told me that his manuscript on 'points proches' (*Weil 1953*) was written for Bourbaki as a first draft of the volume on Differential Geometry, but that Bourbaki finally decided not to use it. Thus, it seemed to me at the time that the subject was ripe for using topos theory and categorical logic.

Apart from an unpublished talk at Ehresmann's seminar in Amiens in 1975 on 'Calcul différentiel à la Fermat', my first published paper was one in collaboration with Gavin Wraith (*Reyes and Wraith 1978*). The advantage of the use of the language was shown convincingly. The axiom $W$ that Wraith had discovered was clearly formulated to define the Lie algebra of a Lie group and the usual properties proved using the internal language. An aside to this construction is that in models (discovered later) these notions go beyond the classical notions defined in terms of manifolds. For instance the Lie algebra of the space of automorphisms of a manifold is a 'smooth' space, although this group is not a manifold in general. Another great advantage of the use of the internal language was the fact that 'functions', 'spaces' and so on became honest to God mathematical notions and thus set-theoretical notations could be freely used.

The context to do SDG is a topos with a commutative ring $R$. Using the internal language we define $D = \{d \in R : d^2 = 0\}$ and write the main axiom as the assertion of ordinary mathematics 'For any function $f : D \longrightarrow R$ and any $x_0 \in R$ there is a unique couple $(a, b)$

Proof only

such that for every $d \in D(f(d) = a + db)$'. Furthermore, any assertion deduced from this axiom using only *constructive* or *intuitionistic logic* is automatically true in the topos. In particular if $f$ is a function defined on some set $U$ closed under addition by an infinitesimal (e.g. an open set) and $x_0 \in U$ then the $f(x_0 + d) = a + db$ (by applying the axiom to the function $\phi(d) = f(x_0 + d)$). Letting $d = 0$ we obtain the truncated Taylor series expansion

$$\forall\, d \in D\, [f(x_0 + d) = f(x_0) + df'(x_0)]$$

This gives us the following recipe: to find $f'(x_0)$ write the equation $f(x_0 + d) = f(x_0) + db$ and solve for $b$ by identifying coefficients of $d$ on both sides.

As a very simple example let $f(x) = \sqrt{x}$. The recipe give us successively

$$\sqrt{x_0 + d} = \sqrt{x}_0 + bd$$
$$x_0 + d = x_0 + 2b\sqrt{x}_0 d \quad \text{(by squaring both sides noticing that } d^2 = 0)$$
$$1 = 2b\sqrt{x}_0 \quad \text{(by identifying coefficients of } d)$$
$$b = 1/2\sqrt{x}_0$$

I have been teaching calculus using this approach with some measure of success to non-mathematics students for the last few years.

Further axioms may be added. For instance we may like to add that $R$ is local. To show the consistency of the axioms, models are required and these appeared at a later date as I mentioned already.

Apart from my paper with G. Wraith already mentioned and a belatedly published paper with *Bélair and Reyes 1985*, my early work on SDG was done mainly in collaboration with Anders Kock. He had clearly formulated and popularized the main axiom of SDG which today is usually called, appropriately enough, axiom of Kock-Lawvere. In particular we were able to clarify notions like n-forms by using infinitesimal structures directly. For instance an n-form on $M$ is a map $M^{D^n} \times D^n \longrightarrow R$ assigning a number (a size, like length, area, volume) to every infinitesimal $n$-cube subject to some conditions (homogeneity, alternation, degeneracy) (Kock and Reyes 1979). The definition of exterior derivative of Elie Cartan via circulation along an infinitesimal parallelogram (*Cartan 1928*) may be taken 'verbatim' and Stokes theorem may be proved nicely in this context. Other notions like parallel transport could also be defined in an illuminating way, thanks to the presence of infinitesimal spaces and a flexible language to describe them and prove things about them. Another application was a proof of a local form of the Gauss-Bonnet's theorem in dimension 2 obtained by adding infinitesimal angles in two different ways. An improved version of this proof as well as the previous results may be found in my book *Models for Smooth Infinitesimal Analysis* (*Moerdijk and Reyes 1991*) in collaboration with Ieke Moerdijk.

I should say that there is another, combinatorial approach to these notions that was pioneered by Joyal and was developed further by Kock. It is based on the notion of points being 1-neighbors or 2-neighbors, etc. Although the field of application is limited essentially to classical manifolds, there are genuine advantages in this approach. Quite recently some algebraic geometers have have taken an interest in this approach. However, this goes beyond the limits of my talk.

## 5. Rear window reflections

Looking back to what we did in the decade of the seventies it is impossible not to reflect on what was achieved and what was not, the possibilities realized and those missed.

I guess that the great lines at least of categorical logic were laid and the work done since then has enriched this framework. In particular the connection with geometry has

Proof only

become deeper. Notions such as open, connected and locally connected have been defined for geometric morphisms between toposes.

But my hopes that the advancement of geometry would result in better techniques to prove logical theorems have not been realized. It is possible that this missed opportunity is still with us. Similarly, and maybe as a consequence, our hopes that logicians working on model theory take up these ideas and techniques and use them to further develop model theory were not realized. Maybe we failed to show clearly the connection between our work and theirs, but I feel that this is was a missed opportunity on their part. Contrary to our expectations, intuitionists took seriously topos theory and used them with profit as the text by Troesltra and van Dalen on 'Constructive Logic' shows (*Troelstra and van Dalen 1988*). Some people in proof theory also used work by Lambek and Lawvere. It seems that some people coming from Computer Science have taken up topos theory and category theory seriously and we can hope new developments from this quarter. My work was addressed mainly to model-theorists but only intuitionists and computer scientists seem to have paid any attention. I cannot restrain myself from remembering Jorge Luis Borges observation that Jonathan Swift with his 'Gulliver's travels' had tried to write a book to condemn humanity but left a book for children instead.

With respect to SDG. A lot has been done and we have at our disposal a language and a theory to make rigorous the calculus of infinitesimals used by geometers, physicists and engineers. As we pointed out with Moerdijk in our book, these people have used not only invertible infinitesimals (which is the subject of Non-standard analysis), but above all nilpotent infinitesimals to deal with notions such as parallel transport, differential forms, curvature and so on.

Although some category theorist have lately pursued SDG (see for instance *Bunge et al. 2018*), no geometers, however, have used SDG to the extent that analysts have used non-standard analysis.

In 1999 I attended a Peripatetic Seminar in Cambridge to celebrate the arrival of Gavin Wraith to the select group of the sexies (called 'sexagenerians' by some envious people). In his talk he observed that the enormous work done on topos theory since Grothendieck and his school had not succeeded in entering the mainstream of mathematics. Whether the future will revert this situation is not for me to tell.

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